BY THE SAME AUTHOR

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Fundamental ideas of functions, limits etc., have been explained in simple and lucid language without lack of rigour. No difficulties appear to have been suppressed or slurred over. At every stage theory has been amply illustrated by worked examples.... A large number of neat and accurate diagrams enhance the value of the book. I believe the book will serve the needs of the teacher in the class-room, and also of the private candidate who is preparing for University Examinations without the guidance of a teacher.

– **Mr. K. B. Madhava**, M.A., A.I.A. (Lond.)
Professor of Mathematics, Mysore University.

The book is well planned and will be a good text-book for the Intermediate students. The get-up is quite good.

– **Dr. C. N. Srinivasiengar**, D. Sc.,
Professor of Mathematics, College of Engineering, Bangalore.

PREFACE

Among the Universities in India, Mysore was, perhaps, the earliest to include the elements of Analytical Geometry and Calculus in the Intermediate syllabus in Mathematitcs and this book is intended primarily to supply a want long felt in the Mysore University. Planned some years ago, the book has attained its present form after undergoing a number of modifications suggested by the author's class room experience of several years.

An attempt has been made in this work to lay proper emphasis on the fundamentals. The locus and its equation have been introduced sufficiently early and the equations to straight lines, circles and other curves are derived as natural applications of the basic analytical formulæ connected with points. The early introduction of the gradient of the join of two points and the conditions for parallelism and perpendicularity renders the solutions to severel problems simple and elegant. The book is illustrated with a large uumber of neat and accurate diagrams which are intended to provide the student with a clear insight and a comprehensive outlook. Alternative solutions to many theorems and problems given in the book will serve to heighten the interest of the reader. Another special feature of the book is the variety and range of problems carefully selected and properly arranged to lead the pupils on from the easy to the more difficult ones. It is hoped that by working these problems the student will acquire both confidence anc proficiency. . . . .

The author is highly thankful to Dr. A. Narasinga Rao, Professor of Mathematics, Annamalai University for his kindness in writing a Foreword for this book.

The author is indebted to three of his former students, Mr. V. K. Doraswamy, who has been of immense help in preparing the MS for the book and in working out the problems and verifying the answers, Messers M. Krishna Murthy, M.A. and V. K. Radhakrishna who rendered willing assistance in the arduous task of correcting the proofs, and to Mr. M. Ramaswamy M. Sc., Lecturer in Mathematics, Intermediate College, Bangalore for going through the MS and offering some useful suggestions.

Any constructive criticism or suggestion for improvement is welcome.

The University.
Mysore,
August 1941.

M. V. JAMBUNATHAN.

NOTE TO THE SECOND EDITION

In response to the wishes expressed by a few teachers, a chapter on the circle, containing mostly the results deduced in the earlier sections, is included in this edition.

May, 1944.

M. V. J.

FOREWORD

Sri M. V. Jambunathan has been kind enough to send me an advance copy of the chapters dealing with the straight line in his book on Elementary Analytical Geometry. I found in them a very full and careful treatment with plenty of illustraive material and alternative modes of approach so as to impress the learner with the inevitability of mathematical results by whatever process they may be obtained. The printing and get-up are good and I wish the book every success.

Annamalainagar,
25—8—1941

A. NARASINGA RAO,
Professor of Mathematics,
Annamalai University and Editor
of THE MATHEMATICS STUDENT.

BY THE SAME AUTHOR

ELEMENTARY TRIGONOMETRY

A Test-book for the Intermediate Classes

This book intended to meet the needs of the Intermediate students of Indian Universities is better than the usual run of such books. The subject has been treated well, the author not having confined himself to the four walls of the syllabuses. Different proofs are given for many of the propositions and a large number of examples has been worked out- There is a good selection of problems, and mnemonics and aids to remember the formulæ are also given. The short historical notes introduced make interesting reading and will serve as a stimulent for the students to study more about the subject.

The get-up of the book is good and the figures neat.

—**The Hindu**, Madras.

.... his ELEMENTARY TRIGONOMETRY ....has been planned so as to serve as a *standard text-book on Trigonometry, comprehensive yet clear and simple.* The book has all the good features which would make it very useful to students as well as teachers in our Colleges.

G. A. SRINIVASAN,
Professor of Applied Mathematics,
Presidency College, Madras.

M. V. SESHADRI & Co.,
BALEPET, BANGALORE CITY

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* * *

Elementary Trigonometry — Elementary Calculus
Elementary Analytical Geometry.

SELECT OPINIONS

The author of the above three text books has met with exceptional success in his aim, namely, to present the subject metter of these books in a clear and simple manner so as to rouse the interest of the pupils in the respective subjects.

The book on Trigonometry begins with an interesting historical note on the contribution of Ancient Indian Mathematicians to Trigonometry. Details are discussed fully in the course of the book and the exposition is throughout lucid. The book abounds in worked examples and contains a good number of exercises.

The same high standard of exposition is kept up in the book on Analytical Geometry which contains a thorough discussion of the subject as applied to straight lines.

The auther is to be particularly congratulated on the able manner in which the idea of a limit has been formulated and discussed in his book on Calculus. The notion of a limit, so fundamental in Calculus and higher mathematics, is usually a stumbling block for the beginner. By varied concrete illustrations the author leads up to the general notion in a clear and easy manner.

The get up of all the three books are attractive and the prices are very moderate.

– THE MATHEMATICS STUENT.

ELEMENTARY
ANALYTICAL GEOMETRY

CHAPTER I

Introduction

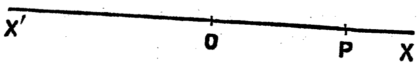
1. The Subject of Analytical Geometry. The chief feature of Analytical Geometry, or Algebraic Geometry, as it is sometimes called, which distinguishes it from the Euclidean and other forms of Pure Geometry is that in the former we introduce algebraic symbols and freely apply algebraic methods in solving geometrical problems. It has been aptly said of Analytical Geometry that, while the subject is geometrical, the language is that of Algebra.

Analytical Geometry, aided by the powerful yet simple methods of Algebra, not only enables us to have a firmer grasp of the subject but also furnishes universal methods for the solution of a number of problems and makes possible a study of further problems not thought of by the earlier geometers, thus widening the boundaries of geometrical knowledge.

The invention of Analytical Geometry is attributed to the renowned French philosopher and mathematician, Rene Descartes, who, in his book "La Geometrie" (1637), showed for the first time how the methods of Algebra could be applied to the study of Geometry. On this account this subject is sometimes called Cartesian Geometry.

In order to apply algebraic methods to the study of Geometry, it is first of all necessary to devise some suitable means of representing the position of a point, line etc. by algebraic quantities. This is dealt with in the following paragraphs.

2. Axis. Co-ordinate. Let X′OX be any straight line of indefinite length, and let O be a fixed point on it. Suppose now, a point P is free to move on the line X′X. The position of P on the line is known if the length OP is known. For definiteness, let P be taken to the right of O. As the point P moves further to the right the length OP *increases*; while, if it moves to the left the length OP *decreases*. We may thus consider the *right* as the *increasing* direction and the *left* as the *decreasing* direction.



(Fig 1)

Note:- It is equally valid to take P to the left of O and argue that the length OP increases as P moves to the left; in order therefore to remove the ambiguity as well as to simplify subsequent work, we may choose *either* of the two and adhere to it throughout, that is to say, we may adopt any *convention*, The present convention is to consider the increasing direction.

Since, if P coincides with O itself, the length OP becomes zero, we infer that for any position of P to the right of O the length OP is *greater than zero*, that is, the length OP is *positive* and for any position of P to the left of O the length OP is *less than zero* or *negative* For this reason, OX is called the positive direction and OX′ the negative direction of the line X′OX.

The position of any point P is then described by giving the measure of OP, i.e the number denoting the distance OP, taken with the proper sign. Thus it is seen that a point P on the line X′OX can be represented by a number, positive or negative or zero. This number is called the **adscissa** or the **co-ordinate** of the point P, while the fixed point O is called the **origin** and the line X′OX is called the **axis.**

3. Directed lines. Steps. Let A, B be two points on an axis, origin O. Then the measure of AB with the positive sign prefixed if the direction of AB, *from* A *to* B coincides with the positive direction of the axis, (as explained in Art. 2) or the negative sign prefixed if the direction of AB coincides with the negative direction of the axis, is called the **directed line-segment AB** or **step AB.** The step AB is sometimes denoted by $\overline{AB}$. It is clear from the adove definition that

$$\text{step } AB = -\text{step } BA$$

or

$$\overline{AB} = -\overline{BA}$$

It is also seen that

$$\text{step } AB + \text{step } BA = 0,$$

or

$$\overline{AB} + \overline{BA} = 0.$$

We may consider the step AB as the act of walking from A to B, or a journey from A to B; the step BA, then, represents a journy from B to A. The resultant effect of these two journeys is to bring the individual back to the starting point, A. This is what is symbolically expressed by the above equation, viz.,

$$\overline{AB} + \overline{BA} = 0.$$

Another important property of steps is that if A, B be any two points on the axis and **O** the origin, then

$$\overline{AB} = \overline{OB} - \overline{OA},$$

or, simply,

$$\mathbf{AB = OB - OA},$$

for all the positions of the origin **O**, the *order of the letters* indicating the direction or sense of each segment.

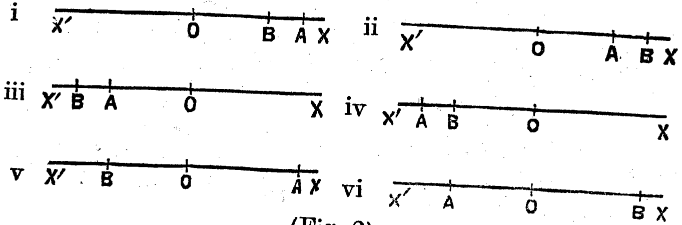
There are six possible cases, as illustrated in Fig. 2

In Fig. 2 (i) $OB = OA + AB.$

$$\therefore\ AB = OB - OA.$$

In Fig. 2 (ii)

$$OA = OB + BA$$
$$= OB - AB$$
$$\therefore AB = OB - OA.$$



(Fig. 2)

The other cases are proved in exactly a similar manner and are left as an exercise for the student.

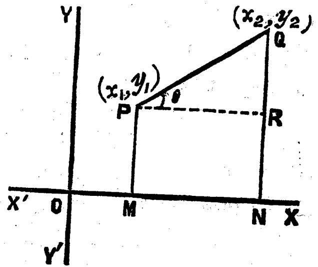
Corollary. If x_1 and x_2 be the co-ordinates of the points A and B respectively, then in all cases

$$AB = x_2 - x_1$$

4. Application to Projection. Let PQ be any directed line segment and X′X, any other line. Let M and N be the *projections* of P and Q respectively on X′X, that is, the feet of the perpendiculars from P and Q on the line X′X. The step MN is called the **projection** of the directed line PQ on the line X′X.

It is to be observed that the step NM is the projection of the segment PQ, whence it follows that

projection of PQ = − projection of QP.

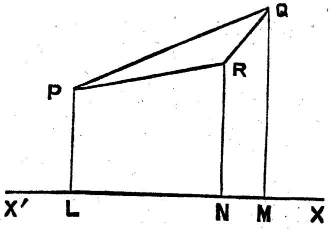


(Fig. 3)

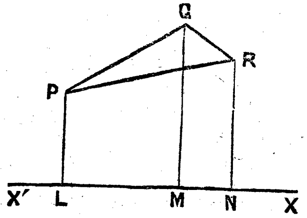
If l should represent the segment PQ and θ the inclination of PQ to X′X, then it is clear that

$$\text{projection of PQ on X'X} = MN = l \cos \theta.$$

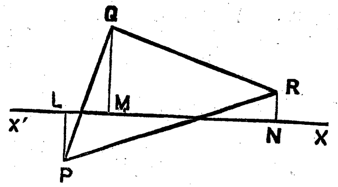
5. Projection of a Broken Line. Let PRQ be any broken line joining the points P and Q and let L,M,N be the projections of P,Q,R respectively on the line X′X. The sum of the projections on X′X of the segments PR, RQ is called the projection of the broken line PRQ. Therefore, the projection on X′X of the broken line PRQ



(Fig. 4)



(Fig. 5)



(Fig. 6)

$$= LN + LM = LM$$

in all the cases.

But LM is the projection of the segment PQ on the line X′X. Thus the projection of the line PQ on X′X is equal to the projection of the broken line PRQ. Similarly it can be shown that the projection of PR on X′X is equal to the projection of any broken line joining P and R. And this process can be repeated any number of times.

Hence we arrive at the following result:

The projection of a directed line-segment PQ *on any line* X′X *is equal to the sum of the projections of the segments of any broken line joining the two points* P *and* Q

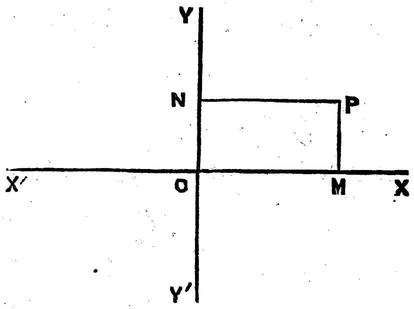
This result is important and finds some useful applications in subsequent work.

An immediate corollary of the above result is the following:

The sum of the projections of PR, RQ and QP is zero; in other words, the sum of the projections of the sides of a triangle taken in order, on any straight line in the same plane, is zero.

CHAPTER II

Co-ordinates of a Point. Loci



(Fig. 7)

6. Cartesian Co-ordinates. Let X′OX, Y′OY be any two perpendicular straight lines meeting at O, and P any point in the plane of these lines. Let PM, PN be the perpendiculars from P on X′X, Y′Y respectively.

The two directed line-segments or steps OM, ON or NP, MP determine completely the position of the point P. The distance of P from the Y-axis i. e. NP or OM is called the **abscissa** or the x **co-ordinate** of P, while the distance of P form the x-axis i. e. MP or ON is called the **ordinate** or the y-**co-ordinate** of P. The abscissa and the ordinate of a point are together referred to as its **co-ordinates.**

The point O is called the **origin,** the line X′OX the x-**axis** and the Y′OY the y-**axis.** The x- and y- axes are together referred to as the **co-ordinate axes** or the **axes of reference.**

Thus, if OM = 4, ON=2, then, with respect to (contracted into *w. r. t*) the axes X′OX and Y′OY, the abscissa of P is 4, the ordinate of P is 2. Or, we might say that the co-ordinates of P are 4, 2. We write this symbolically as P≡(4,2).

(Note that the co-ordinate that is written first always denotes the abscissa.)

In general if OM=x, ON=y, then P is called the point (x, y).

It will be observed that the abscissa is the projection of OP on the x-axis and the ordinate the projection of OP on the y-axis.

The two axes of reference may be inclined to each other at any angle; but for convenience they are taken at right angles.

This system of representing the position of a point by means of its abscissa and ordinate with reference to two co-ordinate axes is called the **Cartesian system,** and the co-ordinates themselves are referred to as the **Cartesian co-ordinates.**

The co-ordinates of a variable point are denoted by (x, y) while those of a fixed point by (x_1, y_1), (x_2, y_2) or (h, k).

Note 1. From the definition of co-ordinates, it follows that the y-co-ordinate of any point on the x-axis is 0, while the x-co-ordinate of any point on the y-axis is 0.

Note 2. Both the co-ordinates of the origin are zero.

Exercise I

1. Where do all points whose abscissa is zero lie ?
2. Where do all points whose ordinate is zero lie ?

7. Signs of Co-ordinates. Quadrants. Adopting a convention similar to that explained in Art. 2, we take OX (Fig. 7) as the positive direction of the x-axis and OX′ as the negative direction. Similarly, OY is taken as the positive direction of the y-axis and OY′ as the negative direction.

The two axes of reference, namely X′OX, Y′OY divide the plane of the paper into four parts, called **quadrants.**

XOY is called the **first quadrant.** YOX′ the **second quadrant,** X′OY′ the **third quadrant** and Y′OX the **fourth quadrant.**

In accordance with the convention of signs stated above it is clear that

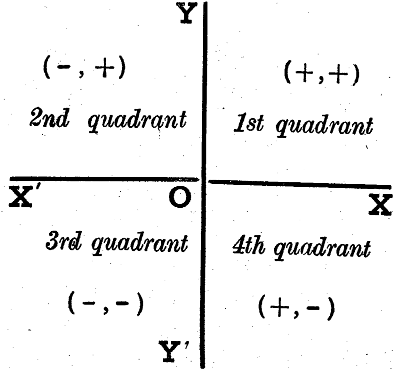
(1) a point in the first quadrant has its abscissa and ordinate both positive,

(2) a point in the second quadrant has its abscissa negative and ordinate positive,

(3) a point in the third quadrant has its absicssa and ordinate both negative and

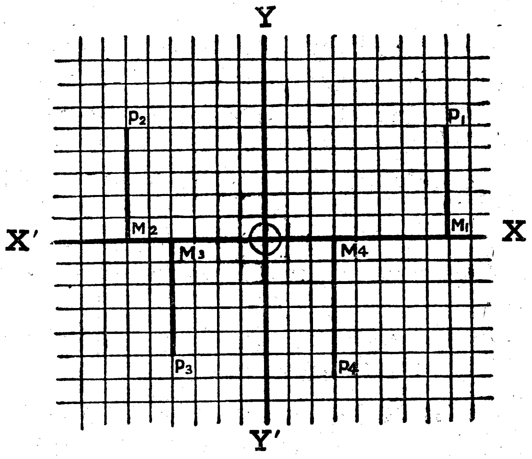
(4) a point in the fourth quadrant has its abscissa positive and ordinate negative.

These results can be expressed diagramatically thus:



8. Plotting of Points. Given the co-ordinates of a point, we may mark in a diagram the position of the point with reference to two perpendicular lines taken as the axes. This process of marking the position of the point is called *plotting.*

For instance, if the co-ordinates of a point be (8, 5), we proceed along OX (Fig. 8) a distance equal to 8 units until we arrive at M_1; at M_1 draw $M_1P_1 \perp$ to OX and equal to 5 units. P_1 is the position of the point (8, 5). Similarly, the points $P_2 \equiv (-6, 5)$, $P_3 \equiv (-4, -5)$, $P_4 \equiv (3, -6)$ are plotted in the diagram.



(Fig. 8)

Exercise II

State the quadrant in which the point lies given that

1. its x-co-ordinate is positive, y co-ordinate negative
2. its abscissa is negative, ordinate negative.
3. its abscissa is negative, ordinate positive.
4. its x-co-ordinate is positive.
5. its y-co-ordinate is positive.
6. its obscissa is negative.
7. its x-and y-co-ordinates are both positive.
8. its abscissa and ordinate have like signs.

9. its abscissa and ordinate have unlike signs
10. Plot the following points on squared paper:
    $(3, 2)$; $(0, -1)$; $(0, 0)$; $(-2, -2)$; $(4, 0)$; $(-2\frac{1}{2}, 0)$; $(1\frac{1}{2}, 1\frac{1}{3})$; $(-3, 1)$.
11. In which quadrants do all points lie whose abscissa and ordinate are (i) equal both in sign and in magnitude?
    (ii) equal in magnitude but opposite in sign?
12. If M be the foot of the ordinate of the point P (h,k) and PM be produced to its own length to Q show that $Q \equiv (h, -k)$.
13. Show that the line joining the points (x, y) and $(-x, -y)$ passes through the origin and is bisected at that point.
14. Show that $(x/2, y/2)$ is the mid-point of the line joining the origin and the point (x, y).

9. Loci. Definition. *The path traced out by a point which moves so that it always satisfies some geometrical law or condition is called the* **locus** *of a point.*

Thus, if the condition be that the moving point is at a constant distance from a fixed point; then the path traced out is a circle, which is, therefore, called the locus of the moving point. Again, if a point is equidistant from two intersecting lines, the locus of a point is the pair of bisectors of the angles between the two lines.

There is another method of defining a locus.

When it is stated that a point satisfies one or more given conditions, it is sometimes found that there is not merely one but an infinite number of points, *all satisfying simultaneously* the given condition. The *aggregate* of such points is called the locus of points satisfying the conditions.

Thus, if the condition be that the distance from a fixed point is constant, there are an infinite number of points lying on the circumference of a circle, all of which satisfy the given condition. All these points on the circle constitute the locus. More briefly, we say that the circle is the locus.

It is to be noted that all points satisfying the given condition lie on the locus and all points on the locus satisfy the given condition, a result which is sometimes expressed thus :

The given condition is satisfied by all the points on the locus and by no point outside it.

10. Equation to a Locus. The geometrical condition satisfied by the points on a locus may also be expressed in the form of a relation satisfied by the co-ordinates of every one of the points. If we consider, for example, the locus of points at a distance of 3 units from the x-axis and above it, we find the locus to be a straight line parallel to the x-axis. All the points on this line satisfy the relation that the y-co-ordinate is 3, or simply, $y=3$. This relation or equation is called the *equation to a locus.* We therefore say that the equation of a straight line || to the x-axis and at a distance of 3 units above it is $y=3$.

In the same manner since the abscissa of all points on the y-axis is 0, we say that the equation to the y-axis is $x=0$. Similarly the equation to the x axis is $y=0$.

Example 1. *What is the locus of points whose co-ordinates satisfy the equation $x=y$?*

Since x and y are equal, the points are equidistant from the two co-ordinate axes; and since x and y are of the same sign, the points should lie in the first and the third quadrants only.

The locus is, therefore, the straight line bisecting the angle XOY and its vertically opposite angle.

Example 2. *What is the equation to the locus of points distant -5 from the y-axis ?*

The distance from the y-axis is the abscissa of the point which by data, must always be -5.

Hence the equation is $x=-5$, or $x+5=0$.

11. Test for a Point lying on a Locus. Let us consider a simple case of a locus, namely, $x=y$.........(1)

It is often necessary to determine whether a point P (h, k) lies on the locus or not. For that we have only to examine whether the abscissa and ordinate of P are connected by the same relation as is expressed by the equation to the locus, namely, $x=y$; in other words, we should test whether the condition $h=k$ is true or not. If it is true, the point P lies on the locus given by (1), otherwise, not.

Hence, to test whether a given point (h, k) lies or not on a locus whose equation is given, substitute for x, y in the given equation the values h, k; if the resulting equation is true, the point (h, k) lies on the locus, otherwise not. We express this by saying that a point lies on a locus when its co-ordinates *satisfy* the equation to the locus.

12. Intersection of two Loci. It is clear from the preceding Article that if a point (h, k) lies on each of two loci the co-ordinates (h, k) should satisfy separately the equations to the two loci. That is, such values of x, y that satisfy simultaneously both the equations give the co-ordinates of the points common to both the loci or the points of intersection of the loci. For instance, if the two loci be $x=y$ and $x+4=0$, the values of x, y that satisfy both the equations are $x=-4$, $y=-4$; hence the two loci intersect at the point $(-4, -4)$.

13. Intersection with the Axes. We may make use of the above result to find where a given locus cuts the co-ordinate axes. The co-ordinates of a point common to the locus $y=x+1$ and the x-axis whose equation is $y=0$ is obtained by solving these two as simultaneous equations. We thus get $x=-1$, $y=0$, which shows that the point of intersection is $(-1, 0)$. Similarly, to find where $y=x+1$ cuts the y-axis, solve the equations $y=x+1$ and $x=0$ obtaining $x=0$, $y=1$; the point of intersection is thus $(0, 1)$

A working rule to find the points in which a locus cuts the x-and y-axes is this;

with the x-axis: Put $y=0$ in the equation to the locus, the resulting value for x gives the abscissa of the point of intersection.

with the y-axis; Put $x=0$ in the equation to the locus, the resulting value for y gives the ordinate of the point of intersection.

Example. Find the co-ordinates of the points at which the two loci $x+y=1$, $2x-5=y$ *meet each other and meet the co-ordinate axes.*

Solving the two equations, we get $x=2$, $y=-1$.

$\therefore$ The two given loci meet at the point $(2, -1)$.

Applying the rule of Art. 13, we find that the locus $x+y=1$ meets the x-axis in $(1, 0)$ and the y-axis in $(0, 1)$; the locus $2x-5=y$ meets the x-axis in $(2\frac{1}{2}, 0)$ and the y-axis in $(0, -5)$,

Exercise III

Find the locus of a point whose co-ordinates (x,y) satisfy the condition.

1. $x=2$ **2.** $y=0$ **3.** $y=-3$ **4.** $2x+5=0$.

5. $x+y=0$ **6.** $2x=y$ **7.** $3x-4y=0$ **8.** $x+5y=0$

Interpret the relations:

9. $x=a$ **10.** $y+b=0$ **11.** $x=2y$ **12.** $x=y+1$

Find the points of intersection of the following pairs of loci:

13. $x=4$, $y=2$ **14.** $x+1=y$, $2x=y$.

15. $y=3x+1$, $x^2+y^2=17$.

16. Find the points at which the following loci cut the co-ordinate axes:

(1) $x+y=3$. (2) $x+y=0$. (3) $x^2+y^2=4$.

(4) $y=x^2-1$. (5) $x=5$.

Find the equation to each of the following loci:

17. The st. line bisecting the angle XOY and the vertically opposite angle.

18. The st. line at a constant distance $+6$ from the x-axis

19. Points which lie on a line through the origin inclined at an angle θ to OX and $90°-\theta$ to OY. (θ acute)

20. Points which lie on a line through the origin inclined at θ to OX and $\theta-90°$ to OY. (θ obtuse)

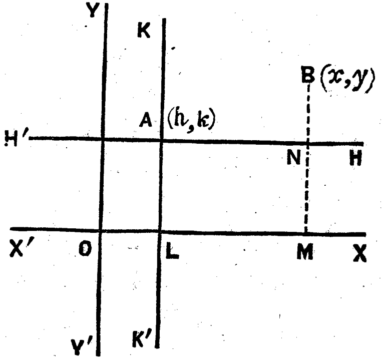
21. A rectangle is constructed so that two of its adjacent sides lie along the co-ordinate axes OX, OY. If the perimeter of the rectangle be constant and equal to $2s$, find the equation to the (i) locus of the vertex opposite O (ii) locus of the point of intersection of the diagonals of the rectangle.

14 Change of Origin. Let A be a point having co-ordinates (h,k) referred to the two perpendicular lines X′OX Y′OY as axes (Fig. 9, 10). Through A draw H′AH, K′AK $\parallel$ to the co-ordinate axes. Let BM be the ordinate of any other point B (x, y). Let H′AH meet BM. produced if necessary, at N and let K′AK meet the x-axis at L.

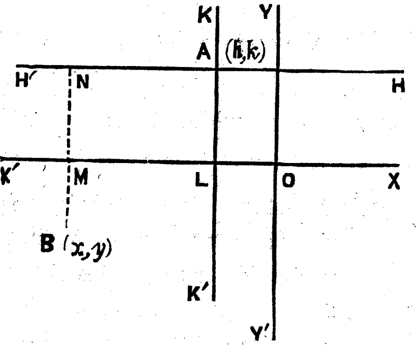
Then, due regard being paid to the signs of the segments,

$OL=h,\ LA=k,$
$OM=x,\ MB=y,$
$AN=LM$
$\quad =OM-OL$
$\quad =x-h,$
$NB=MB-MN$
$\quad =MB-LA$
$\quad =y-k.$

The segment AN, which is the distance of B from A measured parallel to the x-axis, is called the *x-distance* of B from A, and its magnitude as shown above is $x-h$. Similarly, the segment NB, which is the distance of B, from A, measured parallel to the y-axis, is the *y-distance* of B from A and is equal to $y-k$.



(Fig. 9)



(Fig. 10)

Considering A as the origin and H′AH, K′AK as the axes of reference, let us denote the co-ordinates of B by (X, Y); then

$$X=AN=x-h,$$
$$Y=NB=y-k,$$

that is to say, if the origin be shifted from O to the point A (h,k) without changing the directions of the axes, the co-ordinates of the point B (x,y) change to $(x-h,\ y-k)$; in other words, the co-ordinates of any point (x, y) with reference to parallel axes through (h, k) are $(x-h,\ y-k)$.

Conversely, we may obtain x, y given X, Y by using the relations $\quad x=X+h, \quad y=Y+k.$

Example 1. *To what point should the origin be shifted in order that the point* (8, 2) *may have its abscissa halved and ordinate trebled ?*

Let (h, k) be the point to which the origin is to be shifted. The new co-ordinates of the point (8, 2) are given by

$$X=8-h, \qquad Y=2-k.$$

But $\quad X=\frac{1}{2}.8$ and $Y=3.2,$

i.e. $\quad 8-h=4$ and $2-k=6$

$$\therefore h=4, \quad k=-4.$$

Hence the origin is to be shifted to the point (4, −4).

Example 2. *By a suitable shifting of the origin, find the co-ordinates of the mid-point of the line joining the origin to* (x_1, y_1).

Let A $(x_1\ y_1)$ be the given point, O the origin, and P (x,y) the mid-point of OA. Shift the origin to P; the new co-ordinates of O and A become $(-x, -y)$ and (x_1-x, y_1-y) respectively.

Since the new origin is the mid-point of the line joining O, A we have (by Ex. II Q. 13)

$$-x=-(x_1-x), \quad -y=-(y_1-y).$$

whence $\quad x=x_1/2, \quad y=y_1/2.$

$\therefore$ The co-ordinates of the mid-point are $(\frac{1}{2}x_1, \frac{1}{2}y_1)$.

Exercise IV

1. Express the co-ordinates of the point (7, 5) *w. r. t.* parallel axes through (2, 3).

2. What are the co-ordinates of the point (3, - 3) *w. r. t.* parallel axes through (i) (1, 1) (ii) (− 2, 5) (iii) (2, 0) (iv) (½, - 3) (v) (3, - 3) ?

3. Find the point whose co-ordinates when referred to || axes through (- 1, 4) may change to (2, 2)

4. When the co-ordinates of (4, 7) are expressed *w. r. t.* parallel axes through a point on the bisector of the angle XOY, it is found that the abscissa is half the ordinate; find the point.

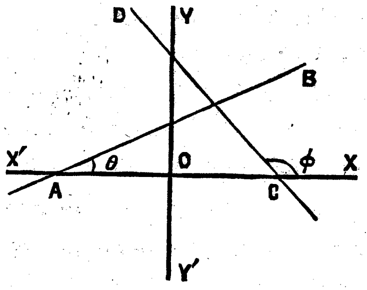
5. To what point should the origin be shifted in order that the point (− 5, 4) may have its abscissa halved and ordinate doubled ?

6. To what point should the origin be shifted in order that the abscissa and ordinate of (- 2,6) may become interchanged ?

7. A(3, 8), B(5, 2) are two given points; when the origin is shifted to a point C, the co-ordinates of A, B become equal in magnitude but opposite in sign. What are the co-ordinates of C *w. r. t.* the original axes ?

15. Inclination of a Line to *X*-axes.

The inclination of a straight line to the x-axis is the angle through which the x-axis should rotate in the positive or anti-clockwise direction, so as to coincide with the given line. Thus in Fig. 11, the inclination of AB to the x-axis is the angle XAB or θ, while that of CD is the angle XCD or ϕ.



(Fig. 11)

It is to be noted that θ is acute and ϕ obtuse.

16. *If* A, B *are the points* (x_1, y_1), (x_2, y_2), *and if the line* AB *be of length* r *and inclined to the x-axis at an angle* θ, *then,*

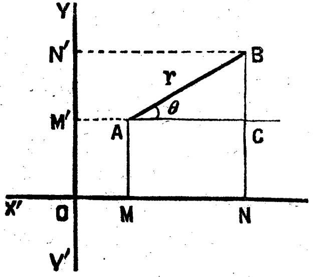
the x-distance of B *from* $A = x_2 = x_1 = r \cos \theta$,

the y-distance of B *from* $A = y_2 - y_1 = r \sin \theta$.

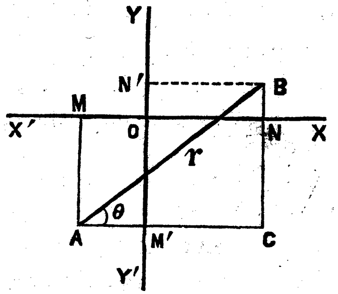
Let AC (Fig. 12, 13) be drawn || to OX cutting NB the ordinate of B, produced if necessary, in C.

The x-distance of B from $A = MN = x_2 - x_1$;

but $MN = AC = r \cos \theta$.



(Fig. 12)



(Fig. 13)

Similarly, the y-distance of B from $A = y_2 - y_1$

$= M'N' = CB$

$= r \sin \theta$, in both the figures.

Note 1· These results are very important and should be remembered.

Note 2. It is to be observed that $MN = (x_2 - x_1)$ and $M'N' = (y_2 - y_1)$ are the projections of AB on the x-axis and the y-axis respectively.

17. Generality of the Results.

The student should have, by now, noticed one characterstic feature of the results obtained in the preceding Articles, namely, that *if* *due regard be paid to the sense of the lines or segments then no matter what the position of the origin is with respect to the points considered the same result is obtained in all cases.*

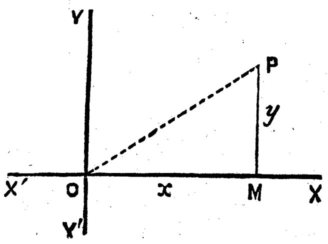
This fact is of great importance, for, it enables us to get a general result, true in all cases, by making use of the simplest and most convenient type of a diagram. There will thus, be no loss of generality, if we take in our diagram the point or points considered in the first quadrant, as we very often do in order to simplify the geometry of the figure, the result deduced being of universal validity.

This is otherwise easily seen to be true; for any finite set of points can be made to lie in the first quadrant by a suitable shifting of the origin.

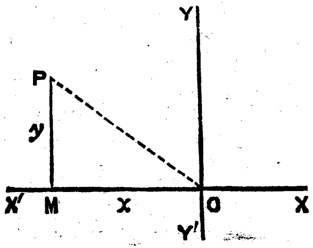
CHAPTER III

Distance Formula

18. Distance of a Point from the Origin. Let P be any point whose co-ordinates are (x, y). P may lie in any quadrant; two of the different cases are shown in Figs. 14 and 15. Let PM be the ⊥ from P to the x-axis; then



(Fig. 14)



(Fig. 15)

$OM = x$, $MP = y$; let $OP = d$. Applying Pythagoras' Theorem to the right angled triangle OMP, we have

$$OP^2 = OM^2 + MP^2.$$

Here we note that, whatever may be the sign of OM or MP, the quantities OM^2, MP^2 are always positive.

Hence, on substitution, whatever the sign of x or y, we get
$$d^2 = x^2 + y^2.$$

Therefore, in whichever quadrant the point (x, y) may lie we have
$$d^2 = x^2 + y^2,$$
or
$$d = \sqrt{(x^2 + y^2)}.$$

Note 1. OP is always considered positive.

Note 2. The four points (x_1, y_1), $(-x_1, y_1)$, $(-x_1, -y_1)$, $(x_1, -y_1)$ are equidistant from the origin.

19. Distance of a point from the origin. Alternative Method. Let $OP = r$ and $\angle XOP = \theta$. (Figs. 14, 15, 16). The student who is familiar with Trigonometry will see that, wherever P may lie,

$$\cos\theta = \frac{OM}{OP} = \frac{x}{r},$$

$$\sin\theta = \frac{MP}{OP} = \frac{y}{r}.$$

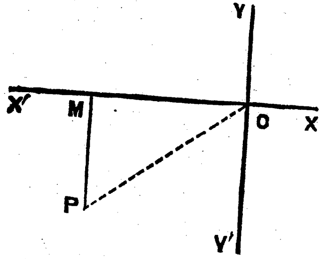
so that $x = r\cos\theta$,

$y = r\sin\theta$.

Squaring and adding,

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2,$$

which gives the distance r as $\sqrt{(x^2+y^2)}$.



(Fig. 16)

Example 1. *Show that the points* O(0, 0), A$(-1\frac{2}{5}, 4\frac{4}{5})$ B(3, 4) *are the vertices of an isosceles triangle, also find the lengths of the equal sides.*

Applying the distance formula, we find that

$$OA^2 = (-1\tfrac{2}{5})^2 + (4\tfrac{4}{5})^2 = 7^2/5^2 + 24^2/5^2 = 25,$$

$$OB^2 = 3^2 + 4^2 = 25.$$

$$\therefore OA = OB = 5.$$

Hence O, A, B are the vertices of an isosceles triangle its equal sides being each equal to 5.

Example 2. *Show that the points* (3,4), $(2\sqrt{6},1)$, $(-4, -3)$ *are equidistant from the origin.*

Denoting the three points by A, B, C and the origin by O, we have

$$OA^2 = 3^2 + 4^2 = 25,$$

$$OB^2 = (2\sqrt{6})^2 + 1^2 = 25,$$

$$OC^2 = (-4)^2 + (-3)^2 = 25,$$

$$\therefore OA = OB = OC = 5.$$

Hence A, B, C are equidistant from the origin.

Exercise V

Find the distance of each of the following points from the origin:

1. (3, 4) **2.** (4, -3) **3.** (7, 1) **4.** (-12, -35)
5. (-20,21) **6.** (-3, $2\frac{1}{2}$) **7.** ($-1{\cdot}5$, -2) **8.** ($2{\cdot}8$, $-4{\cdot}5$)
9. ($\frac{12}{13}$, $\frac{5}{13}$) **10.** ($-6\frac{3}{5}$, $11\frac{1}{5}$) **11.** ($\sqrt{12}$, 2) **12.** (3, $-\sqrt{5}$)

Show that the following sets of points are equidistant from the origin:

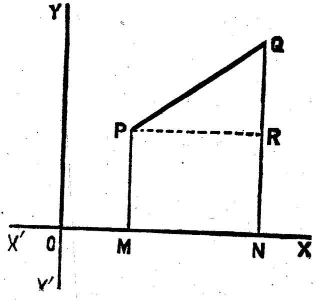
13. (24, -7), (-15, 20), (7, 24).

14. (-16, 63), (25, 60), (56, 33).

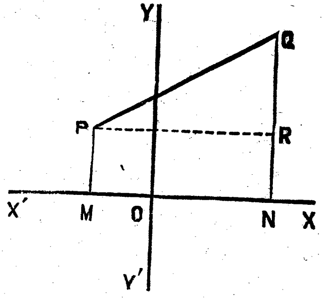
15. Find the abscissa of a point whose ordinate is 15 and distance from the origin 17.

16. Find the ordinate of a point whose abscissa is -9 and distance from the origin 41.

17. Find a point whose abscissa is three-fourths of its ordinate and whose distance from the origin is 35.



(Fig. 17)



(Fig. 18)

20. Distance between two Points. Let P Q be any two given points (Figs. 17, 18) having co-ordinates (x_1, y_1), (x_2, y_2) respectively. Draw PM, QN the ordinates of P, Q and draw PR ‖ to the x-axis to meet QN, produced if necessary in R.

Then, OM$=x_1$, ON$=x_2$, MP$=y_1$, NQ$=y_2$.

Also PR$=$MN$=$ON$-$OM$=x_2-x_1$,

due regard being paid to the signs of the segments;

and RQ$=$NQ$-$NR (taken with their proper signs),

$=$NQ$-$MP$=y_2-y_1$

That is, the two sides of the right angled triangle PRQ are x_2-x_1 and y_2-y_1. Denoting the hypotenuse PQ by d and applying Pythagoras' Theorem, we find

$$d^2=\text{PQ}^2=(x_2-x_1)^2+(y_2-y_1)^2,$$

or

$$d=\sqrt{(x_3-x_1)^2+(y_2-x_1)^2}.$$

Note 1. The result deduced above may be remembered easily in the form

$$(\text{Actual distance})^2=(x\text{-distance})^2+(y\text{-distance})^2,$$

since the x-distance is (x_2-x_1) and the y-distance (y_2-y_1).

Note 2. It is to be observed that d^2 may also be expressed as $(x_1-x_2)^2+(y_1-y_2)^2$; in other words, (x_1, y_1) may be taken as the co-ordinates of either of the points P or Q.

Note 3. Putting $x_2=0$, $y_2=0$ in the formula deduced above, we get the distance between (0, 0) and $(x_1\ y_1)$ as $\sqrt{(x_1^2+y_1^2)}$. Thus, the result of Art. 18 is seen to be a particular case of the result of this Article.

21. Distance between two Points. Alternative Method. Taking as before, the two points as (x_1, y_1), (x_2, y_2), shift the origin to (x_1, y_1), keeping the directions of the axes the same. Let the co-ordinates of the point (x_2, y_2) *w. r. t.* these new axes be (X, Y); then, as in Art. 14.

$$X=x_2-x_1;\quad Y=y_2-y_1$$

∴ The distance, d of this point from the new origin is given by the relation

$$\begin{aligned} d^2&=X^2+Y^2\\ &=(x_2-x_1)^2+(y_2-y_1)^2. \end{aligned}$$

22. The distance between two points may also be derived by another method.

Denoting the distance PQ by r, we note that the projection of r on the x-axis is (x_2-x_1), and that on the y-axis is (y_2-y_1). But, if θ be the inclination of PQ to OX, the projections are by Art. 4, $r\cos\theta$ and $r\sin\theta$.

$$\therefore\ r\cos\theta=x_2-x_1.\quad r\sin\theta=y_2-y_1.$$

Squaring and adding, $r^2=(x_2-x_1)^2+(y_2-y_1)^2$.

Example 1. *Show that the points* (5, −7) *lies on the perpendicular bisector of the line joining* (3, 4) *and* (−5, −2).

The distance between (5, −7) and (3, 4) is

$$\sqrt{\{(5-3)^2+(-7-4)^2\}}=\sqrt{(2^2+11^2)}=\sqrt{125};$$

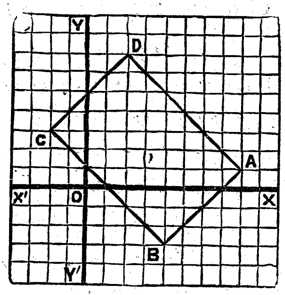
the distance between (5, −7) and (−5, −2) is

$$\sqrt{\{(5+5)^2+(-7+2)^2\}}=\sqrt{(10^2+5^2)}=\sqrt{125}.$$

∴ The point (5, −7) is equidistant from the points (3, 4), (−5, −2) and hence lies on the perpendicular bisector of the line joining these two points.

Example 2. *Show that the points* (8, 1), (4, −3), (−2, 3) (2, 7) *are the vertices of a rectangle and find the lengths of its diagonals.*

It is desirable to have a rough sketch of the figure, as shown in the adjoining diagram.



(Fig. 19)

Denoting the given points in order by A, B, C, D we have

$$\begin{aligned} &\text{AB}^2=4^2+4^2=32,\\ &\text{BC}^2=6^2+6^2=72,\\ &\text{CD}^2=4^2+4^2=32,\\ &\text{DA}^2=6^2+6^2=72,\\ &\text{AC}^2=10^2+2^2=104, \end{aligned}$$

∴ AB$=$CD, BC$=$DA;

hence ABCD is a parallelogram.

Also AB^2+BC^2$=32+72=104=$AC^2;

hence $\angle ABC$ is a right angle.

$\therefore$ ABCD is a rectangle.

Each diagonal of the rectangle is $\sqrt{104}$ or $2\sqrt{26}$.

Example 3. *Find the co-ordinates of a point equidistant from the three points* $(3, 1)$, $(-1, 4)$, $(1, -2)$

Let (x, y) be the co-ordinates of the point. The distances of (x, y) from each of the given points are equal, and hence the squares of the distances are equal.

That is $\quad (x-3)^2+(y-2)^2=(x+1)^2+(y-4)^2$
$\qquad\qquad\qquad\qquad\qquad\quad =(x-1)^2+(y+2)^2,$

or $\quad x^2-6x+9+y^2-4y+4=x^2+2x+1+y^2-8y+16,$

and $\quad x^2-6x+9+y^2-4y+4=x^2-2x+1+y^2+4y+4.$

Transposing, $\quad 8x-4y+4=0\,; \quad 4x+8y-8=0\,;$

solving for x and y we get $\quad x=0, \quad y=1.$

Therefore the co-ordinates of the point equidistant from the three given points are $(0, 1)$.

Note:- The point equidistant from the three points is the circumcentre of the triangle formed by them.

Exercise VI

1, Find the distance between each of the following pairs of points:

1. $(1, 3), (13, 8)$. 2. $(2, -1), (9, 23)$
3. $(-2, 3), (17, 11)$. 4. $(4, -5\tfrac{1}{2}), (7\tfrac{1}{2}, -1\tfrac{1}{2})$
5. $(0{\cdot}7, -0{\cdot}5), (-1{\cdot}3, 16)$.
6. $(2\sqrt{2}\ -1).\ (-\sqrt{2}, -1+\sqrt{31})$.

2. Show that the points $(0, 0), (-7, 24), (15, 20)$ are the vertices of an isosceles triangle and find the length of its base

3. Show that the points $(7, 3), (5, 5), (-1, -3)$ are the vertices of an isosceles triangle and calculate the perimeter of the triangle.

4. Show that the point $(-6, 7)$ lies on the perpendicular bisector of the line joining $(2, 3)$ and $(-2, -1)$.

5. Find the sides of the triangle whose vertices are $(-\sqrt{3}, 0), (1-\sqrt{3}, -2), (0, \sqrt{3})$ and hence find the angles of the triangle.

6. If $A \equiv (-1, -5)$, $B \equiv (1, -1)$, $C \equiv (6, 9)$, find the distance AB, BC, AC and hence show that A, B, C are collinear.

7. Show that the point $(1, 1)$ bisects the join of $(-2, -3)$ and $(4, 5)$.

8. Show that $(1, -1), (-1, 1), (\sqrt{3}, \sqrt{3})$ form an equilateral triangle.

9. Show that $(1, 4), (-4, -1), (2, 1)$ are the vertices of a right-angled triangle. Find the vertex at which the right angle is formed.

10. Show that $(-7, 3), (4, 5)$ subtend a right angle at $(-4, -1)$ and compute the area of the triangle formed by the three points.

11. Prove that $(4, 1), (1, 6), (-4, 3), (-1, -2)$ are the vertices of a square. Find the area of the square.

12. Show that the points $(3, 2), (-2, 2), (-2, -2)$ and $(3, -1)$ form a rectangle and find the length of its diagonals.

13. Show that $(-2, 0), (2, 1), (4, 2)$ and $(0, 1)$ form a parallelogram; also find the ratio of the lengths of its diagonals.

14. Show that $(-1, 3), (-4, -5), (5, -7)$ and $(8, 1)$ are the vertices of a parallelogram and find the lengths of the diagonals.

15. Show that the points $(3, -2), (7, 6), (-1, 2), (-5, -6)$ are the vertices of a rhombus having one of its diagonals three times the other. Also calculate the area of the rhombus.

16. Show that the points $(-2, -1)$, $(-1, 6)$, $(-5, 3)$ are the vertices of a square and find the fourth vertex.

17. Show that the points $(7, -1)$, $(1, -1)$ and $(4, 3)$ are the vertices of a rhombus. Find the fourth vertex and the ratio of the diagonals.

18. If $(2, 7)$ and $(-6, 3)$ be one pair of opposite vertices of a square, find the other pair of opposite vertices.

19. Prove that $(-2, 2)$, $(-1, 1)$, $(2, 4)$ are the vertices of a rectangle and find the fourth vertex.

20. Show that $(-7, 24)$, $(-15, -20)$, $(24, 7)$, $(20, -15)$ all lie on a circle whose centre is the origin. What is the radius of this circle?

21. Prove that the points $(-2, -2)$, $(-1, 1)$, $(7, 1)$, $(8, -2)$ all lie on a circle whose centre is at $(3, -2)$. Also find the radius of this circle.

22. Find the co-ordinates of a point equidistant from the three points $(3, 2)$, $(0, 5)$, $(-5, 2)$.

23. Find the circum-centre and circum-radius of the triangle whose vertices are

(1) $(8, 4)$, $(3, -1)$, $(7, 7)$ (2) $(1, 1)$, $(1, 2)$, $(-1, 3)$

(3) $(2, 4)$, $(-2, 0)$, $(4, -2)$ (4) $(2, -3)$, $(3, -1)$, $(-1, -2)$

24. Show that the four points $(4, 0)$, $(6, 1)$, $(4, 3)$, $(3, 2)$ are concyclic. Also find the centre of the circle.

25. Prove that the four points $(-4, 2)$, $(1, 5)$, $(-1, -3)$, $(4, 0)$ lie on a circle. Find the centre and radius of the circle.

23. Application to Locus Problems. We have already seen in Art. 10 that the geometrical condition satisfied by the points on any locus can be expressed as an equation connecting the co-ordinates of these points and that this equation is referred to as *the equation to the locus*.

Making use of the formula for the distance between two points it is possible to determine the equations to several loci which are of common occurrence. Thus, if it be given that a point is at a distance of 8 units from the origin, we know from Geometry that the locus is a circle whose centre is at the origin and radius 8 units. But, if the point have co-ordinates (x, y), its distance from the origin is $\sqrt{(x^2+y^2)}$ which by data is equal to 8 units; that is to say, the geometrical condition when expressed analytically becomes

$$\sqrt{(x^2+y^2)}=8 \quad \text{or} \quad x^2+y^2=64.$$

This equation connecting the co-ordinates of any point on the locus is, as has already been explained, spoken of as the equation to the locus. The equation therefore, to the circle whose centre is at the origin and radius 8 units is

$$x^2+y^2=64.$$

24. General Equation to a Circle. If (x, y) be the co-ordinates of any point on a circle whose centre is at (h, k) and radius r, the equation satisfied by (x, y) is

$$(x-h)^2+(y-k)^2=r^2,$$

or

$$x^2+y^2-2hx-2ky+(h^2+k^2-r^2)=0 \ldots\ldots\ldots\ldots(1)$$

The student should note that in this equation to the circle

the co-efficient of x^2=the co-efficient of y^2,

the xy-term is absent.

If we are given the equation

$$x^2+y^2+2gx+2fy+c=0 \ldots\ldots\ldots\ldots(2)$$

(where g, f, c are constants), it may be written in the form

$$(x+g)^2+(y+f)^2=(g^2+f^2-c),$$

which expresses the fact that the square of the distance of (x, y) from the fixed point $(-g, -f)$ is equal to a constant quantity, (g^2+f^2-c). The locus is, therefore, a circle whose radius is $\sqrt{(g^2+f^2-c)}$ and whose centre is at the point $(-g, -f)$.

The equation may also be given in the form

$$Ax^2+Ay^2+2Gx+2Fy+C=0, \ldots\ldots\ldots\ldots(3)$$

which, on division by A, reduces to

$$x^2+y^2+2(G/A)x+2(F/A)y+C/A=0$$

which is of the same form as equation (2). It can then be easily shown that (3) represents a circle whose centre is at the point $(-G/A, \ -F/A)$ and radius $\sqrt{(G^2+F^2-AC)}/A$,

Example 1. *Find the equation to the circle whose centre is* (2, 1) *and radius* 4.

Let (h, k) be any point on the circle.

The square of the distance of the point from the centre (2, 1) expressed analytically is $\quad (h-2)^2+(k-1)^2$.

Since by data the distance is 4, the (distance)^2=16.

$$\therefore (h-2)^2+(k-1)^2=16.$$

The equation satisfied by the co-ordinates of any point on the circle is $\quad (\text{abscissa}-2)^2+(\text{ordinate}-1)^2=16.$

i. e. $\quad (x-2)^2+(y-1)^2=16.$

In other words the equation to the circle, centre (2, 1) and radius 4 is $\quad (x-2)^2+(y-1)^2=16.$

Example 2. *Find the centre and radius of the circle.*

$$2x^2+2y^2+7x-4y+8=0.$$

Divide the given equation throughout by 2,

$$x^2+y^2+7x/2-2y+4=0.$$

This can be expressed in the form

$$(x+7/4)^2+(y-1)^2=1/16=(1/4)^2,$$

which is seen to be the equation of a circle, centre at $(-7/4, 1)$ and radius 1/4. Hence, the centre of the given circle is $(-7/4, 1)$ and radius 1/4.

Example 3. *Find the equation to the circum-circle of the triangle whose vertices are* (3, 6), (–1, 4) (–1, –2).

Let (h, k) be the circum-centre; equate the distance of the three given points from (h, k):

$$(h-3)^2+(k-6)^2=(h+1)^2+(k-4)^2=(h+1)^2+(k+2)^2,$$

i.e. $\quad h^2-6h+9+k^2-12k+36=h^2+2h+1+k^2-8k+16$

$$=h^2+2h+1+k^2+4k+4.$$

Transposing, $\quad 8h+4k-28=0,$

$$8h+16k-40=0;$$

solving for h and k, we get $\quad h=3, \ k=1.$

The circum radius = distance of (3, 1) from any of the given points = 5.

∴ Equation to the circle, centre (3, 1) and radius 5 is

$$(x-3)^2+(y-1)^2=25.$$

which is the equation required.

25. Further Applications of Distance Formula to Locus Problems. It has alreay been explained in Art. 23 that it is possible by making use of the distance formula, to determine the equations to several loci. We have already dealt with the case in which the distances considered happen to be equal. We work below a few problems wherein the distances are related in other ways.

Example 1. A, B *are the points* (1, 0), (4, 0); P *is a variable point such that* PB = 2PA; *find the eqation to the locus of* P.

Taking the co-ordinates of P as (x, y), we have

$$PA^2=(x-1)^2+y^2,$$

$$PB^2=(x-4)^2+y^2.$$

But $\quad PB^2=4PA^2;$

$$\therefore (x-4)^2+y^2=4\{(x-1)^2+y^2\}$$

i.e. $\quad x^2-8x+16+y^2=4x^2-8x+4+4y^2;$

transposing and simplifying $\quad x^2+y^2=4.$

Hence the equation to the locus of P is $x^2+y^2=4$, which is easily seen to be a circle.

It may be noted that this circle is referred to in Geometry as the "Appollonius' Circle."

Example 2. *Find the equation to the locus of a point* P *which moves so that its distance from the point* (2, 3) *is equal to its distance from the axis of* y.

Taking the co-ordinates of P as (x, y) the distance of P from (2, 3) is $\sqrt{(x-2)^2+(y-3)^2}$ and that from the y-axis is x. By data, these distances are equal, and therefore the squares of these distances are equal,

i.e. $$(x-2)^2+(y-3)^2=x^2.$$

Simplifying, the equation to the locus of P is obtained as

$$y^2-4x-6y+13=0.$$

Exercise VII

1. Find the equation to the perpendicular bisector of the line joining (5, −1) and (−1, 5). Does the bisector pass through the origin ?

2. Write down the equations to the following circles :

(1) centre (0, 0), radius 2 (2) centre (4, 0), radius 7,

(3) centre $(-3, 1\frac{1}{2})$ radius $3\frac{1}{2}$

(4) centre (−1, 1), radius $\sqrt{5}$

(5) centre (1, 2) and passing through (3, 4)

(6) passing through (3, 1), (2, 2) and radius 1.

3. Find the centre and radius of the circle whose equation is

(1) $x^2+y^2-2x-4y-4=0$ (2) $x^2+y^2+4x-6y-12=0$

(3) $x^2+y^2+4x+8y+11=0$ (4) $4(x^2+y^2)+5x-8y+4=0$

4. Using the fact that any diameter of a circle subtends a right angle at any point on the circumference, derive the equation to the circle described on the line joining (1, 2), (3, 8) as diameter.

5. Following the method of Q. 4, show that the equation to the circle described on the join of (x_1, y_1), (x_2, y_2) as diameter is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.

6. Find the equation to the circle passing through the points (11, 3), (−1, −15), (−13, −7).

7. Find the equation to the circle circumscribing the triangle formed by (0, 0), (1, 2), (3, 4).

8. Find the equation to the locus of a point which is such that its distance from (1, 0) is one-third of its distance from (9, 0).

9. Find the equation to the locus of a point whose distances from (0, 8), (0, 2) are in the ratio of 2 : 1.

10. Find the equation to the locus of a point which moves so that its distance from the point (2, 1) is three times its distance from (−1, 3).

11. The distance of a moving point from (3, 0) is equal to its distance from the axis of y. Find the equation to its locus.

12. The distance of a point from the origin is four times its distance from the axis of x ; find the equation to its locus.

13. The sum of the squares of the distances of a point P from the points $(a, 0)$, $(-a, 0)$ is equal to a constant quantity $2c^2$. Find the equation to the locus of P.

14. Find the equation to the locus of a point, the difference of whose distances from (0, 5) and (0, −3) is equal to 4.

15. A straight line of constant length moves so as always to have its two ends on the two axes of reference ; find the equation to the locus of the mid-point of the line.

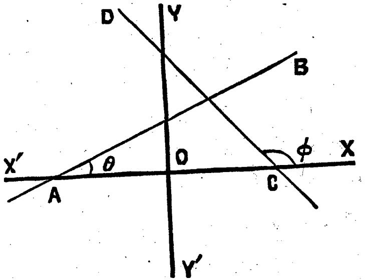
16, A circle cuts the co-ordinate axes so as to form chords of constant lengths $2a$ and $2b$ respectively. Show that the equation to the locus of the centre is

$$x^2-y^2=a^2-b^2.$$

CHAPTER IV

Gradient

26. Gradient of a Straight Line. It has been explained in Art. 15 that the inclination of a line to the x-axis is the angle through which the x-axis should rotate in the anti-clock-wise direction so as to coincide with the given line. Thus in the adjoining figure, the inclination of AB to the x-axis is the angle XAB or θ, while that of CD is the angle XCD or ϕ.



(Fig. 20)

The tangent of the inclination of a given line to the x-axis is called the **slope** or the **gradient** of the line. Thus the slope of AB (Fig. 20) is $\tan\theta$ and the slope of CD is $\tan\phi$.

Since θ is acute and ϕ abtuse, it is important to remember that $\tan\theta$ is positive and $\tan\phi$ negative.

The slope of the line AB is, therefore, positive; on this account we say that AB is inclined positively to the x-axis.

The slope of CD is negative and therefore we say that CD is inclined negatively to the x-axis.

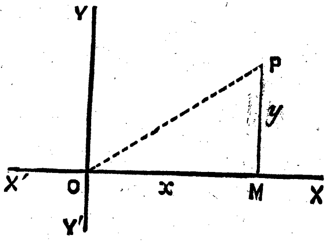
The slope of a line is usually denoted by the letter m. Therefore m, slope, gradient, $\tan\theta$, tangent of the inclination to the x-axis — all these are used in the same sense.

Corollary. It is to be noted that since parallel lines are equally inclined to the x-axis, the tangents of these angles are equal; in other words, the slopes of parallel lines are equal. Conversely, lines having equal slopes are parallel.

27. Slope of the Line joining a Point to the Origin.

If P(x, y) be a point such that OP is inclined at an angle θ to the x-axis, it is easily seen that

slope of OP $= \tan\theta = y/x$.



(Fig. 21)

This is otherwise obvious from the fact that

$x = r\cos\theta,\ y = r\sin\theta,$

where r is the distance OP;

$\therefore$ by division $\tan\theta = y/x$.

Note. Any line || to the y-axis is inclined at 90° to the x-axis; the slope such a line, therefore, ∞.

Example. Find the inclination to the x-axis of the line joining (2, 5) *to the origin.*

The slope of the line is $\frac{5}{2}$ or $2\frac{1}{2}$; i.e. $\tan\theta = 2\frac{1}{2}$, where θ is the inclination of the line to the x-axis.

$\therefore\ \theta = 68°\ 12'$, (from tables)

Exercise VIII

Find the slopes of the following lines:

1. A line inclined at 45° to the x-axis
2. A line inclined at 60° to the x-aixs.
3. A line inclined at 150° to the x-axis.
4. The x-axis.
5. The y-axis.
6. A line parallel to the x-axis.
7. The line $x=3$.
8. The line $y+4=0$.
9. The bisectors of the angles between the co-ordinate axes.

10. Find the inclination to the x-axis of a line whose slope is (1) 0. (2) 1. (3) -1. (4) $-\sqrt{3}$. (5) $\frac{1}{2}$. (6) $-\frac{3}{4}$. (7) $2+\sqrt{3}$. (8) $\sqrt{3}-2$.

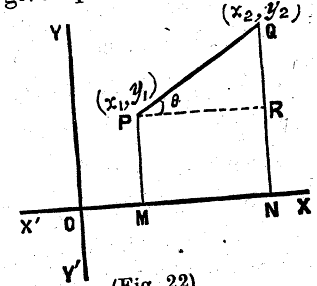
11. Find the slopes of the line joining the origin to the following points ;

(1) $(2, 2)$. (2) $(-3, 6)$. (3) $(-1, -3)$. (4) $(-2, 0)$.

12. Find the inclination to the x-axis of the line joining the origin of the point (1) $(3, 3)$. (2) $(4, 0)$. (3) $(0, -2)$. (4) $(-5, 12)$, (5) $(\sqrt{3}, 1)$.

28. Slope of the Line joining two Points.

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be any two given points (Fig. 22). Let PM, QN be the ordinates of P and Q respectively and let PR be drawn ∥ to the x-axis meeting NQ at R.



(Fig. 22)

[The student should draw different figures for the several positions of the points P and Q.]

The inclination of PQ to the x-axis is $\angle QPR = \theta$, say.

$\therefore$ The slope of PQ $= \tan\theta = RQ/PR$.

But $PR = MN = ON - OM = x_2 - x_1$,
$RQ = NQ - NR = NQ - MP = y_2 - y_1$;

hence the slope of $PQ = \dfrac{y_2 - y_1}{x_2 - x_1}$ or $\dfrac{y_1 - y_2}{x_1 - x_2}$.

Corollary 1. The slope of the line joining (x_1, y_1), (x, y) is $(y - y_1)/(x - x_1)$.

2. If (x, y) is any point collinear with (x_1, y_1), (x_2, y_2) then the slope of the line joining (x, y) to (x_1, y_1) is equal to the slope of the line joining (x_1, y_1) to (x_2, y_2), that is

$$(y - y_1)/(x - x_1) = (y_1 - y_2)/(x_1 - x_2).$$

29. Aliter. If in Fig. 22, PQ be of length r, we have

$r\cos\theta =$ projection of r on the x-axis $= x_2 - x_1$,
$r\sin\theta =$ projection of r on the y-axis $= y_2 - y_1$;
$\therefore \tan\theta = (y_2 - y_1)/(x_2 - x_1)$.

Example 1. *Find the slope of the line joining the points* $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$.

$$\text{The slope is} = \frac{2at_1 - 2at_2}{at_1^2 - at_2^2} = \frac{2a(t_1 - t_2)}{a(t_1^2 - t_2^2)} = \frac{2}{t_1 + t_2}.$$

Example 2. *Show that the line joining* $(1, 2)$ *and* $(3, 5)$ *is* ∥ *to that joining* $(-3, -2)$ *and* $(-1, 1)$

$$\text{Slope of the first line} = \frac{5 - 2}{3 - 1} = \tfrac{3}{2}.$$

$$\text{Slope of the second line} = \frac{1 - (-2)}{-1 - (-3)} = \tfrac{3}{2}.$$

The Slope of the two lines are equal; $\therefore$ they are parallel.

Example 3, *Show that* $(-1, -1)$, $(5, 3)$ $(1, 5)$, $(-5, 1)$ *are the vertices of a parallelogram.*

Let the four points be A, B, C, D respectively.

[The student is advised to plot the points to know the order of the vertices.]

Slope of $AB = (3 + 1)/(5 + 1) = \frac{4}{6} = \frac{2}{3}$,
Slope of $CD = (5 - 1)/(1 + 5) = \frac{4}{6} = \frac{2}{3}$,
Slope of $BC = (5 - 3)/(1 - 5) = 2/(-4) = -\frac{1}{2}$,
Slope of $DA = (1 + 1)/(-5 + 1) = 2/(-4) = -\frac{1}{2}$.

$\therefore$ AB is ∥ to CD and BC is ∥ to DA.

Hence ABCD is a parallelogram.

Example 4. A, B, C, *are the points* $(1, 3)$, $(4, 7)$, $(-5, -5)$ *show that* A, B, C *are collinear.*

$$\text{Slope of AB} = \frac{7 - 3}{4 - 1} = \frac{4}{3}, \quad \text{slope of AC} = \frac{3 + 5}{1 + 5} = \frac{8}{6} = \frac{4}{3}.$$

AB, AC have equal slopes.

$\therefore$ ABC is a st. line, i.e. A, B, C are collinear.

Exercise IX

Find the slopes of the line joining the pairs of points :

1. $(1, 1)$, $(5, 5)$. **2.** $(-1, 2)$, $(7, 6)$.

3. $(a \cos \theta, a \sin \theta)$, $(a \cos \phi, a \sin \phi)$.

4. $(a \cos \theta, b \sin \theta)$, $(a \cos \phi, b \sin \phi)$.

5. Show that the line joining $(-2, 1)$, $(1, 4)$ is ∥ to the line joining $(0, 0)$, $(2, 2)$.

6. A, B, C, D are the points $(-2, 0)$, $(4, 3)$, $(1, -1)$, $(5, 1)$; show that AB is ∥ to CD.

7. P, Q, R, S have co-ordinates $(-1, 3)$, $(1, 0)$, $(4, 2)$, $(0, 8)$ respectively, prove that the parallel through P to RS passes through Q.

8. Show that the points $(1, 2)$, $(5, -2)$, $(3, -6)$, $(-2, -4)$ form a trapezium.

9. Show that the points $(-5, 1)$, $(3, 7)$, $(5, 5)$, $(-1, -3)$ are the vertices of an isosceles trapezium, (i.e. a trapezium in which the non-parallel sides are equal).

10. A, B, C, D are the points $(3, 1)$, $(2, 0)$, $(0, 1)$, $(1, 2)$; show that AB is ∥ to CD and that BC is ∥ to AD. Hence show ABCD is a parallelogram.

11. Prove that $(-3, 2)$, $(5, 4)$, $(-1, -1)$, $(3, 7)$ are the vertices of a parallelogram.

12. Show that $(2, 0)$, $(-2, 1)$, $(-4, 2)$, $(0, 1)$ form a parallelogram having one of its diagonals parallel to the x-axis.

13. Show that the points $(1, 1)$, $(4, 5)$, $(-2, -3)$ lie on one and the same st. line.

14. Prove that $(5, -7)$, $(-3, -1)$, $(-7, 2)$ are collinear.

15. Show that the line joining $(7, -3)$ and $(-5, 1)$ passes through $(-2, 0)$.

16. If (x, y) is any point on the line passing through $(1, 2)$, $(3, 4)$, show that $y=x+1$.

17. Show that the equation to the locus of points collinear with the points $(-1, 2)$, $(-3, 4)$ is $x+y=1$.

18. Show that, for any values of k and l,

$$\left(\frac{kx_2+lx_1}{k+l}, \frac{ky_2+ly_1}{k+l}\right)$$

is collinear with the points (x_1, y_1), (x_2, y_2).

30. Slopes of Perpendicular Lines.

Let AB and CD be two perpendicular lines making angles θ and θ' respectively with the x-axis. (Fig. 23).

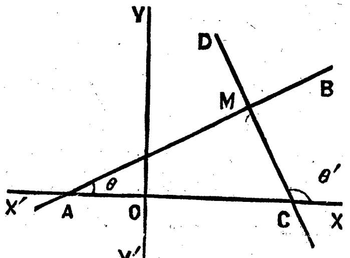
It is easily seen that

$$\theta' - \theta = 90^\circ$$

or $\theta' = 90^\circ + \theta$.

If the slopes of the two lines be m, m', we have

$$m = \tan \theta$$
$$m' = \tan \theta'.$$



(Fig. 23)

But $\tan \theta' = -\tan(180^\circ - \theta')$

$$= -\tan \text{MCA} = -\tan(90^\circ - \text{MAC}) = -\tan(90^\circ - \theta)$$
$$= -\cot \theta = -1/\tan \theta = -1/m,$$

i.e. $m' = -1/m$ or $\mathbf{mm' = -1}$.

Thus, if two straight lines are at right angles, the product of their slopes $= -1$.

Conversely, it can be proved by retracing the steps that if $mm' = -1$, the two st. lines are at right angles.

Example 1. *Show that the points* $(4, 3)$, $(-1, -2)$, $(1, 4)$ *are the vertices of a right-angled triangle and find the vertex containing the right-angle.*

Let the points taken in order be A, B, C.

Slope of $\text{AB} = (3+2)/(4+1) = 1$,
slope of $\text{BC} = (4+2)/(1+1) = 3$,
slope of $\text{AC} = (3-4)/(4-1) = -\frac{1}{3}$.

∴ AC and BC are at right angles since the product of their slopes is -1.

∴ ABC is a triangle right-angled at the vertex C (1, 2).

Example 2. A, B, C, D *are the points* $(-2, 3)$, $(6, 7)$, $(2, -1)$, $(-6, -5)$; *show that the figure* ABCD *is a rhombus.*

$$\text{Slope of AB} = \frac{7-3}{6+2} = \tfrac{1}{2}, \qquad \text{slope of BC} = \frac{7+1}{6-2} = 2,$$

$$\text{slope of CD} = \frac{-1+5}{2+6} = \tfrac{1}{2}. \qquad \text{slope of DA} = \frac{3+5}{-2+6} = 2.$$

$$\text{slope of AC} = \frac{3+1}{-2-2} = -1, \qquad \text{slope of BD} = \frac{7+5}{6+6} = 1.$$

Thus, AB, CD have equal slopes and are therefore ∥
BC, DA have equal slopes and are therefore ∥.

∴ ABCD is a parallelogram.

The diagonals AC, BD are at right angles since the product of their slopes is -1, while the adjacent sides AB, BC are not at right angles since the product of their slopes $\neq -1$.

Hence the figure ABCD is a rhombus.

Example 3. *Find the equation to the circle whose diameter is the line joining the points* $A(x_1, y_1)$ *and* $B(x_2, y_2)$.

Let $P(x, y)$ be any point on the circle; then AB subtends a rt. angle at P, i.e. PA, PB are at rt. angles. The condition for this is that the product of the slopes of PA, PB is -1.

But the slope of PA is $\dfrac{y-y_1}{x-x_1}$ and that of PB is $\dfrac{y-y_2}{x-x_2}$

$$\therefore \frac{y-y_1}{x-x_1} \cdot \frac{y-y_2}{x-x_2} = -1;$$

cross-multiplying and transposing.

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0,$$

which is the equation required.

Note. Compare this method with that suggested in Ex. VII, Q. 5

Exercise X

1. Prove that the line joining $(-2, 1)$, $(4, 4)$ is ⊥ to that joining $(0, 1)$, $(2, -3)$.

2. Show that the join of $(1, 5)$, $(-2, 3)$ is ⊥ to the join of $(2, 6)$, $(4, 3)$.

3. A, B, C, D are the points $(1, 2)$, $(4, -1)$, $(3, -2)$, $(-2, -4)$; prove that the diagonals of the quadrilateral intersect at right angles.

4. Show that the points $(3, 4)$, $(-2, -1)$, $(4, 1)$ are the vertices of a right-angled triangle.

5. A, B, C are the points $(4, 1)$, $(-1, -4)$, $(1, 2)$; prove that the triangle ABC is right angled and that the hypotenuse is $5\sqrt{2}$.

6. Show that the points $(-4, 3)$, $(1, 6)$, $(7, -4)$, $(2, -7)$ are the vertices of a rectangle.

7. Show the $(1, 2)$, $(4, 5)$, $(5, 4)$, $(2, 1)$ from a rectangle of area 6 sq. units.

8. Show that the points $(1, 4)$, $(6, 1)$, $(3, -4)$, $(-2, -1)$ are the vertices of a square.

9. Show that $(1, 5)$, $(9, 9)$, $(5, 1)$, $(-3, -3)$, from a rhombus.

10. Show that the points $(-5, -19)$, $(-12, 5)$, $(8, 20)$, $(15, 4)$ are the vertices of a rhombus of area 585.

11. Show that $(-1, 6)$ lies on the circle whose diameter is the line joining the points $(3, -2)$, $(-5, 4)$.

12. A, B are the points $(-1, 2)$, $(3, 7)$ and a semi-circle is described on AB; if (x_1, y_1) be any point on the semi-circle show that $x_1^2+y_1^2-2x_1-9y_1+11=0$.

13. Prove that the equation to the circle whose diameter is the line joining $(-2, -3)$, $(5, 4)$ is $x^2+y^2-3x-y-22=0$.

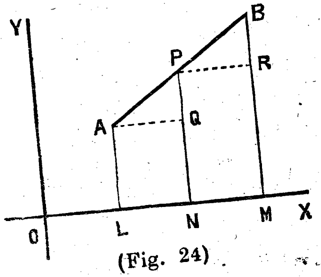
14. Show that if (x_1, y_1), (x_2, y_2) are the extremities of a diagonals of a rectangle, the equation to the circle circumscribing the rectangle is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.

CHAPTER V

Section Formula

31. Point of Bisection of the Line joining two Points.

Let A (x_1, y_1), B (x_2, y_2), be two given points (Fig 24) and let P (x, y) be the point of bisection of AB. Let AL, BM, PN be the ordinates of A, B, P respectively.



(Fig. 24)

It is easily proved that PN bisects LM, i. e. LN=NM.

Further, OL$=x_1$, OM$=x_2$,

ON$=x$;

but OL+OM=2ON,

i. e. $x_1+x_2=2x$.

or $x=(x_1+x_2)/2$.

Also NP = *half* the sum of the parallel sides LA, MB,

i.e. $y=(y_1+y_2)/2$.

Hence the co-ordinates of P are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

32. Aliter.

In Fig. 24 let AQ, PR be drawn || to the x-axis to meet NP, NB at Q, R respectively.

The two right angled triangles AQP, PRB are congruent;

$$\therefore \text{AQ}=\text{PR}, \quad \text{QP}=\text{RB}.$$

But AQ=LN$=x-x_1$, PR=NM$=x_2-x$,

QP=NP - NQ$=y-y_1$, RB=MB - MR$=y_2-y$.

$$\therefore x-x_1=x_2-x, \quad \text{or} \quad 2x=x_1+x_2;$$

$$\therefore x=\tfrac{1}{2}(x_1+x_2).$$

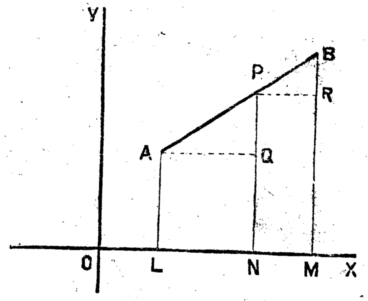
Similarly, $y-y_1=y_2-y$, whence $y=\frac{1}{2}(y_1+y_2)$.

Note. The above result could also be established by the method of projections, and this is left as an exercise for the student.

33. Point dividing the Join of two Points in a given Ratio.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ be any two given points

and let $P(x, y)$ divide AB *internally* in the ratio $k ; l$ (Fig. 25). Draw through A, P parallels to the x-axis to cut the ordinates of P, B in Q, R respectivly.



(Fig. 25)

The right-angled triangles AQP, PRB are easily seen to be similar;

$$\therefore \frac{\text{AQ}}{\text{PR}}=\frac{\text{QP}}{\text{RB}}=\frac{\text{AP}}{\text{PB}}=\frac{k}{l}.$$

But AQ$=x-x_1$, PR$=x_2-x$,

QP$=y-y_1$, RB$=y_2-y$;

$$\therefore \frac{x-x_1}{x_2-x}=\frac{k}{l},$$

so that $lx-lx_1=kx_1-kx$,

or $(k+l)x=kx_2+lx_1$,,

$$\therefore x=\frac{kx_2+lx_1}{k+l}.$$

Similarly, $\frac{y-y_1}{y_2-y}=\frac{k}{l}$, whence $y=\frac{ky_2+ly_1}{k+l}$.

Therefore the co-ordinates of the point which divides internally the join of (x_1, y_1), (x_2, y_2) in the ratio $k:l$ are

$$\left(\frac{kx_2+lx_1}{k+l}, \frac{ky_2+ly_1}{k+l}\right).$$

34. Point of Section. Alternative Method.

If AB is inclined to the x-axis (Fig. 25) at an angle θ, considering the projections of AP, PB on the x-axis, we have

$$\text{AP} \cos\theta=\text{AQ}=x-x_1,$$

$$\text{PB} \cos\theta=\text{PR}=x_2-x,$$

and therefore $(x-x_1)/(x_2-x)=\text{AP}/\text{PB}=k/l$,

so thet $lx-lx_1=kx_2-kx$,

or $(k+l)x=kx_2+lx_1$,

$$\therefore x=(kx_2+lx_1)/(k+l).$$

Similarly, considering the projections on the y-axis, we have $(y-y_1)/(y_2-y)=AP/PB=k/l$, whence, simplifying, $y=(ky_2+ly_1)/(k+l)$.

Note. The student should carefully note and remember that in the above formulæ, k multiplies x_2 (or y_2) and **not** x_1 (or y_1); similarly l multiplies x_1 (or y_1); i e. each number multiplies the co-ordinates of the point that is *away* from the corresponding segment.

Corrolary 1. As an immediate corolary, we find, by putting $k=l$, that the co-orninates of the mid-point of the st. line joining (x_1, y_1), (x_3, y_2) are $\{\frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2)\}$, a result already established in Art. 30,

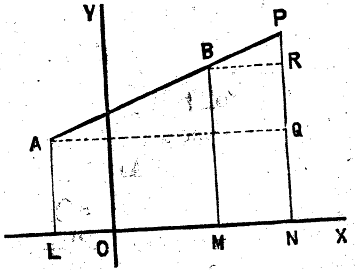
2. Any point (x, y) lying on the line joining (x_1, y_1), (x_2, y_2) divides the line in the ratio

$$(x-x_1)/(x_2-x) \quad \text{or} \quad (y-y_1)/(y_1-y).$$

35. Point of Section : External.

In the preceeding Arts, we considered the point of section to be *internal*; now we take an *external* point of section.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ be two given points (Fig. 26) and let $P(x, y)$ divide AB *externally* in the ratio $k:l$. If the inclination to the x-axis of the line AB be θ, we have



(Fig. 26)

$$AP\cos\theta=AQ=x-x_1,$$
$$BP\cos\theta=BR=x-x_2,$$
$$\therefore \frac{x-x_1}{x-x_2}=\frac{AP}{BP}=\frac{k}{l};$$
$$\therefore lx-lx_1=kx-kx_2$$
$$\text{or} \quad (k-l)x=kx-kx_2,$$
$$\therefore x=\frac{kx_2-lx_1}{k-l}.$$

In a similar manner

$$(y-y_1)/(y-y_2)=k/l,$$

so that $ly-ly_1=ky-ky_2$,

or $(k-l)y=ky_2-ly_1$

whence $h=(ky_2-ly_1)/(k-l)$.

$\therefore$ The point dividing externally the line joining (x_1, y_1), (x_2, y_2) in the ratio $k:l$ is $\left(\frac{kx_2-lx_1}{k-l}, \frac{ky_2-la_1}{k-l}\right)$.

36. General Section Formula.

The results deduced in the two cases of internal and external division may be expressed by a single formula thus :

The co-ordinates of a point which divides the join of (x_1, y_1), (x_2, y_2) in the ratio[1] $k:l$ are given by

$$x=\frac{kx_2+lx_1}{k+l}, \quad y=\frac{ky_2+ly_1}{k+l},$$

where the ratio $k:l$ is considered positive in the case of internal division and negative in the case of external division.

Example 1. *Find the co-ordinates of the points which divide the join of* (4, 3) *and* (1, 6) *internally and externally in the ratio* 1 : 2,

Point of internal division :

$$\text{abscissa}=\frac{1.1+2.4}{1+2}=3, \qquad \text{ordinate}=\frac{1.6+2.3}{1+2}=4.$$

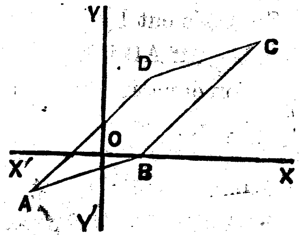
Point of external division .

$$\text{abscissa}=\frac{1.1-2.4}{1-2}=7, \qquad \text{ordinate}=\frac{1\cdot6-2.3}{1-2}=0.$$

$\therefore$ The points of divisiou are (3, 4), (7, 0).

Ex. 2. *Show that* $A(-2,-1)$, $B(1, 0)$, $C(4, 3)$, $D(1, 2)$ *form a parallelogram.*

The mid-point of the diagonal AO (Fig. 27) is (1, 1); and the mid-point of the diagonals BD is (1, 1). Thus, the diagonals AC, BD have the same mid-point, i.e. the diagonals AC, BD bisect each other at (1, 1). Hence ABCD is a parallelogram.



(Fig. 27)

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[1] The case $k:l=-1$, is, obviously, to be excluded.

Example 3. *Find the co-ordinates of the centroid of the triangle whose vertices are* $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$.

If D be the mid-point of BC, its co-ordinates are

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right).$$

The centroid, G, divides AD internally in the ratio 2 : 1; therefore the co-ordinates of G are given by

$$x=\frac{2\times\frac{x_2+x_3}{2}+1\times x_1}{2+1}; \quad y=\frac{2\times\frac{y_2+y_3}{2}+1\times x_1}{2+1},$$

i.e. $x=\frac{1}{3}(x_1+x_2+x_3)$, $x=\frac{1}{3}(y_1+y_2+y_3)$.

$\therefore$ The co-ordinates of the centroid or

$$\{\tfrac{1}{3}(x_1+x_2+x_3),\ \tfrac{1}{3}(y_1+y_2+y_3)\},$$

Example 4. *Find in what ratio the join of* $A(-3, 2)$ *and* $B(4, 6)$ *is cut by the co-ordinate axes.*

Let AB be cut by the x-axis in the ratio $1 : l$; the point of division has co-ordinate $\left(\frac{4-3l}{1+l}, \frac{6+2l}{1+l}\right)$.

But, since this point lies on the x-axis, its y-co-ordinate is 0.

i.e. $(6+2l)/(1+l)=0$ or $6+2l=0$.

$$\therefore\ l=-3.$$

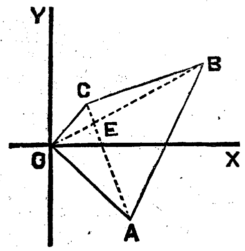
So AB is cut by the x-axis in the ratio $1 : -3$.

Similarly, if AB is cut by the y-axis in the ratio $1 : m$, we have, on equating the x-co-ordinate of the point of division to zero, $\frac{4-3m}{1+m}=0$ or $4-3m=0$,

$$\therefore\ m=\tfrac{4}{3}.$$

$\therefore$ AB is cut by the y-axis in the ratio $1 : \frac{4}{3}$ or $3 : 4$.

Example 5. *Find the ratios in which the diagonals of the quadrilateral* (0, 0), (2, −2), (4, 2), (1, 1) *cut each other.*



(Fig. 28)

Let OABC (Fig. 28) be the quadrilateral whose diagonals intersect at E; let $OE : EB = l : 1$, $AE : EC = m : 1$,

Since E divides OB in the ratio $l : 1$, its co-ordinates are

$$\left(\frac{4l}{l+1}, \frac{2l}{l+1}\right).$$

Also, since E divides AC in the ratio $m : 1$, its co-ordinates are

$$\left(\frac{m+2}{m+1}, \frac{m-2}{m+1}\right).$$

Hence, if $l+1\neq 0$, $m+1\neq 0$, we have

$$\frac{4l}{l+1}=\frac{m+2}{m+1} \text{ and } \frac{2l}{l+1}=\frac{m-2}{m+1}.$$

From the first of these equalities we get

$$4l(m+1)=(l+1)(m+2),$$

or $$3lm+2l-m-2=0.$$

$$\therefore\ m=-2(l-1)/(3l-1).$$

From the second equality we get $2l(m+1)=(l+1)(m-2)$,

or $$lm+4l-m+2=0,$$

$$\therefore\ m=-2(2l+1)/(l-1).$$

Equate these two values of m and cross multiply,

$$(l-1)^2=(2l+1)(3l-1),$$

or $$5l^2+3l-2=0,$$

i.e. $$(5l-2)(l+1)=0,$$

$$\therefore\ l=2/5, \text{ since } l+1\neq 0,$$

$$\therefore\ m=6.$$

Hence the diagonals OB, AC divide each other in the ratios 2 : 5 and 6 : 1.

Alternative Method. Assuming that E divides OB in the ratio $l : 1$, we find the co-ordinates of E to be

$$\left(\frac{4l}{l+1}, \frac{2l}{l+1}\right).$$

But the points A, E, C are collinear; the condition for this (Art. 28, Cor. 2) is that

$$\frac{2l/(l+1)-1}{4l/(l+1)-1}=\frac{1+2}{1-2},$$

i. e.
$$\frac{l-1}{3l-1}=-3 (\text{if } l+1\neq 0);$$

∴ The abscissa of E is $\frac{8}{5}/(1+\frac{2}{5})=\frac{8}{7}$.

But if E divides AC in the ratio $m:1$, the abscissa of E is $(m+2)/(m+1)$.

Hence, if $m+1\neq 0$, we may equate the two, thereby getting

$$(m+2)/(m+1)=\frac{8}{7},$$

$$\therefore\ m=6.$$

Thus the diagonals divide each other in the ratio 2 : 5 and 6 : 1.

Exercise XI

1. Find the co-ordinates of the middle point of the join of (1) (2, 3), (4, 5) (2) (7, − 4), (− 3, 2) (3) (− 6, 0), (8, − 3) (−4, −8); find

2. A, B, C are the points (2, 3), (7, − 5), (− 4, −8); find the co-ordinates of the mid-points of AB, AC; and prove that the line joining these mid-points is ∥ to BC.

3. A (5, 2), B (1, 3), C (− 2, − 1) are the vertices of a triangle; find the mid-points of AB, AC and show that the join of the mid-points is ∥ to BC and equal to $\frac{1}{2}$ BC.

4. A (x_1, y_1), B (x_2, y_2), C (x_3, y_3) are the vertices of a triangle ABC. Prove that the line joining the mid-points of AB, AC is ∥ to BC.

5. Find the co-ordinates of the points which divide internally and externally in the ratio 2 : 1 the line joining the points (7, − 5), (−2, 1).

6. Show that the origin is a point of trisection of the line joinining the points (− 3, 2), (6, −4) and find the other point of trisection.

7. A (−2, 2), B (5, 4), C (3, 1), D (− 4, −1) are four points; show that AC, BD bisect each other and find the co-ordinates of the point of bisection.

8. Show that the points (− 1, 0), (5, 4), (1, 6), (− 5, 2) are the vertices of a parallelogram.

9. Show that the points (3, 1), (− 2, 2), (− 1, 5), (4, 4) are the vertices of a parallelogram.

10. Find the centroid of the triangles formed by

(1) (3, 2), (− 4, 5), (− 8, − 4). (2) (− 1, 4), (3, −7), (−2, 3).

(3) (4, − 4), (5, 0), (− 3, 7). (4) (− $\frac{7}{2}$, 1), (6, $\frac{3}{2}$), (− $\frac{9}{2}$, $\frac{5}{2}$).

11. A, B, C are the points (x_1, y_1), (x_2, y_2), (x_3, y_3); D is the mid-point of BC and G divides AD in the ratio 2 : 1; find the co-ordinates of G and hence deduce that the medians of the triangle ABC are concurrent.

12. A (x_1, y_1), B (x_2, y_2), C (x_3, y_3) are the vertices of a triangle and D, E, F are the mid-points of BC, CA, AB respectively. Show that the centroid of DEF coincides with that of ABC.

13. Show that the lines joining the mid-points of the pairs of opposite sides and the mid-points of the diagonals of a quadrilateral are concurrent and bisect one another.

14. ABC is a tringle in which AB : AC=3 : 7; if (− 1, − 3), (5, 1) be the co-ordinates of B, C, find the points at which the internal and external bisectors of the angle A meet BC.

15. Show that the x-axis divides the join of (1, − 5), (3, 2) in the ratio 5 : 2. Also find the point of division.

16. Find the rations in which the join of (4,1) and (7, − 2) is cut by the co-ordintes axes Find also the points of section

17. Show that the point (7, 5) divides the join of the two pairs of points (1, 2), (5, 4) and (– 5, – 1), (3, 3) in the same ratio; find this ratio.

18. Find the ratios in which the line joining (4, – 3), (– 2,1) and that joining (– 5, – 4), (7, 2) divide each other.

19. Find the ratios in which the join of (– 2, – 1), (8, 5) and the join of (– 2, 2), (3, – 1) divide each other.

20. A (1, 3), B (2, – 5), C (– 4, – 7), D (– 3, 10) are the vertices of a quadrilateral; find the ratios in which E, the point of intersection of the diagonals AC and BD, divides them.

21. A (x_1, y_1), B (x_2, y_2), C (x_3, y_3) are the vertices of a triangle; show that the co-ordinates of the in-centre of the triangle are

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right),$$

where a, b, c denote the sides oppsite A, B, C respectively. Obain similar results for the co-ordinates of the ex-centres.

CHAPTER VI

Areas

37. Area of the Triangle with a Vertex at Origin.

Let O be the origin and A (x_1, y_1), B (x_2, y_2) be any two given points (Fig. 29) and let AL, BM be the ordinates of A. B. It is easily seen that

$$\triangle OAB = \triangle OMB + \text{trapezium BMLA} - \triangle OLA$$

$$= \tfrac{1}{2}OM \cdot MB + \tfrac{1}{2}(MB+LA)ML - \tfrac{1}{2}OL \cdot LA,$$

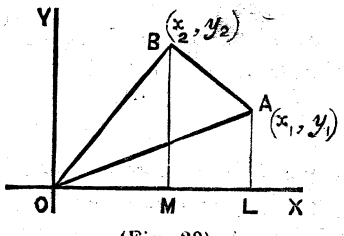
[since area of a trapezium $= \frac{1}{2}$(sum of parallel aides) × (perpendicular distance between them)].

$$= \tfrac{1}{2}[x_2y_2 + (y_1+y_2)(x_1 - x_2) - x_1y_1] = \tfrac{1}{2}(x_1y_2 - x_2y_1).$$

38. Aliter.

Taking OA, OB as r_1, r_2 and their respective inclinations to the x-axis as θ_1, θ_2 we find (as in Art. 19) that

$$x_1 = r_1 \cos \theta_1,$$
$$y_1 = r_1 \sin \theta_1,$$
$$x_2 = r_2 \cos \theta_2,$$
$$y_2 = r_2 \sin \theta_2,$$



(Fig. 29)

$$\therefore \triangle OAB = \tfrac{1}{2}OA.OB.\sin AOB = \tfrac{1}{2}r_1r_2 \sin(\theta_2 - \theta_1)$$
$$= \tfrac{1}{2} r_1r_2(\sin \theta_2 \cos \theta_1 - \cos \theta_2 \sin \theta_1)$$
$$= \tfrac{1}{2}(r_1 \cos \theta_1 . r_2 \sin \theta_2 - r_2 \cos \theta_2 . r_1 \sin \theta_1)$$
$$= \tfrac{1}{2}(x_1y_2 - x_2y_1).$$

Note. The latter method holds good for all positions of the points A and B. The result, of course, is universolly true; *vide* Art. 17.

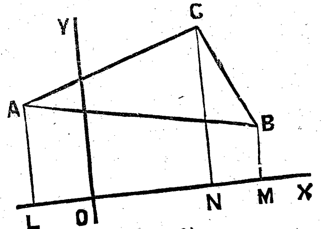
39. Sign of the Area.

If the points be taken in the order BA we obtain by the above formula, the area of the triangle OBA as $(x_2y_1 - x_1y_2)/2$ which is equal to $-(x_1y_2 - x_2y_1)/2$. That is, the area of $\triangle$OBA, is equal in magnitude but opposite in sign to that of the area of $\triangle$OAB. From this we are led to see that the area of a triangle is capable of being positive or negative according as the vertices are taken in the counter-clockwise or clockwise order. We therefore adopt the convention that the area is to be considered positive when the vertices are taken in the positive i.e. counter-clockwise order and negative when the vertices are taken in the negative or clockwise order.

Another method of exprsesing this convention of signs for areas is that, if in going round the figure the area is always to the left, the area is considered positive; if the area is always to the right, the area is considered negative.

40. Area of Triangle with given Vertices.

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be any three given points and O the origin (Fig. 30). The area of the triangle ABC may be derived in a variety of ways.



(Fig. 30)

1. If AL, BM, CN be the ordinates of A, B, C we have

$$\triangle ABC = \text{trapazium ALNC} + \text{trapezium CNMB} - \text{trapezium ALMB}$$
$$= \tfrac{1}{2}(y_1+y_3)(x_3-x_1) + \tfrac{1}{2}(y_2+y_3)(x_2-x_3) - \tfrac{1}{2}(y_1+y_2)(x_2-x_1)$$

which on simplification, given the area required.

Note. This argument is valid only when the x-axis does not cut the tsiangle ABC; but the result is valid in all cases. (See Art. 17)

2. The origin may be shifted to any vertex, say (x_1, y_1) and the co-ordinates of B, C then change to (x_2-x_1, y_2-y_1), (x_3-x_1, y_3-y_1); now applying the formula of Art. 37,

$$\triangle ABC = \tfrac{1}{2}\{(x_2-x_1)(y_3-y_1) - (x_3-x_1)(y_2-y_1)\};$$

the simplification is left as an exercise for the student.

3. Bearing in mind the convention of signs, (Art. 39) it is easily seen that, for all positions of A, B and C.

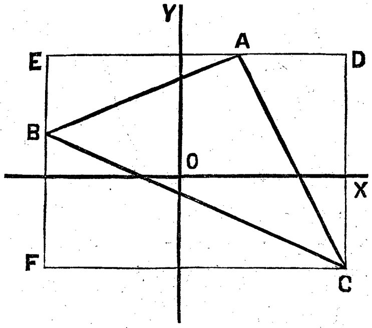
$$\triangle ABC = \triangle OAB + \triangle OBC + \triangle OCA$$
$$= \tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)].$$

an expression that can be re-arranged in the form

$$\tfrac{1}{2}[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)].$$

The student should verify that this agrees with the result given by either of the two methods suggested above.

4. Throught A and C (Fig. 31) draw ||s to the x-axis and through B and C draw ||s to the y-axis, so as to form the rectangle CDEF, in other words, circumscribe about the triangle ABC, a rectangle whose sides are || to the axes.



(Fig. 31)

Now $\triangle ABC = \text{rect. CDEF} - \triangle ACD - \triangle BAE - \triangle CBF.$

$$= CD \cdot FC - \tfrac{1}{2}CD \cdot AD - \tfrac{1}{2}BE \cdot EA - \tfrac{1}{2}FB \cdot FC,$$
$$= (y_1-y_3)(x_3-x_2) - \tfrac{1}{2}(y_1-y_3)(x_3-x_1)$$
$$- \tfrac{1}{2}(y_1-y_2)(x_1-x_2) - \tfrac{1}{2}(y_2-y_3)(x_3-x_2);$$

$$\therefore \triangle ABC = \tfrac{1}{2}[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)].$$

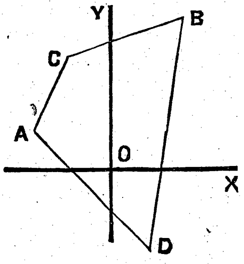
Note. The expression for the area may also be written as

$$\tfrac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2)$$

or as $\tfrac{1}{2}[x_1(y_1-y_3) - y_1(x_2-x_3) + (x_2y_3 - x_3y_2)],$

Example 1. *Find the area of the quadrilateral formed by* $(-2, 1)$, $(2, 4)$, $(-1, 3)$, $(1, -2)$.

It is first necessary to plot the given points to know the order of the vertices. Denoting the four given points in order by A,B,C,D it is seen on plotting (Fig. 32) that the quadrilateral is ADBC.



(Fig. 32)

Area ADBC $= \triangle ADB + \triangle ABC$

$$= \tfrac{1}{2}(12+3+6) + \tfrac{1}{2}(-2+4+3) = 13.$$

Example 2. *Show that the points* $(-3, -1)$, $(-1, 2)$, $(3, 8)$ *are collinear.*

The area of the triangle formed by the three points is

$$\tfrac{1}{2}[(-3)(2-8) + (-1)(8+1) + 3(-1-2)]$$
$$= \tfrac{1}{2}(18 - 9 - 9) = 0.$$

$\therefore$ The three points are such that the area of the triangle formed by them is zero; that is, the three points are collinear.

Example 3. A, B *are the points* $(-1, 3)$, $(-2, 5)$; *find the locus of* $P(x, y)$ *which is such that the area* ABP *is* $+4$.

The Analytical expression for the area of a triangle ABP is $\tfrac{1}{2}[-(5-y) - 2(y-3) + x(3-5)] = \tfrac{1}{2}(1 - 2x - y)$.

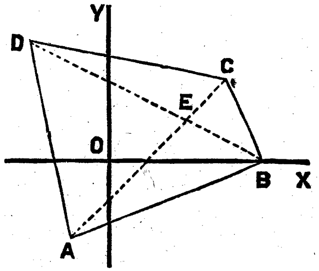
Equate this to the given value of the area namely, $+4$ and simplify; we get $1 - 2x - y = 8$ or $2x + y + 7 = 0$, which is the equation to the locus of P.

Note. On geometrical considerations it will be clear to the student that the locus of P is a st. line || to BC.

Example 4, $A(-1, -2)$, $B(4, 0)$, $C(3, 2)$, $D(-2, 3)$ are *the vertices of a quadrilateral taken in order. Find the ratios in which the diagonals of the quadrilateral are divided at their point of intersection E. Find also the co-ordinates of E.*

It is proved in text-books on Geometry that, if two triangles stand on the same base their area are as the segments into which the line joining their vertices is divided by their common base. Thus, in Fig. 33,

$$\triangle ABD : \triangle BCD = AE : EC,$$
$$\triangle ABC : \triangle ACD = BE : ED.$$



(Fig 33)

Calculating the areas of these four triangles, we have

$\triangle ABD = \tfrac{1}{2}(3+20+4) = 13{\cdot}5$, $\triangle BCD = \tfrac{1}{2}(-4+9+4) = 4{\cdot}5$,

$\triangle ABC = \tfrac{1}{2}(2+16-6) = 6$, $\triangle ACD = \tfrac{1}{2}(1+15+8) = 12$.

$\therefore AE : EC = 3 : 1$ and $BE : ED = 1 : 2$;

$\therefore$ the diagonals are divided at E in the ratios 3 : 1 and 1 : 2.

Also the co-ordinates of E which divides BD in the ratio 1 : 2 are easily obtained as (2, 1).

Note. Compare this method with that employed in Art. 36, *Ex.* 5.

Exercise XII

1. Find the area of the triangles whose vertices are

(1) $(0, 0)$, $(3, 4)$, $(1, 2)$. (2) $(-1, 3)$, $(0, 0)$, $(1, 2)$.
(3) $(-1, -2)$, $(3, 4)$, $(4, 3)$. (4) $(1, 3)$, $(2, 6)$, $(3, 4)$.
(5) $(4, 5)$, $(1, 4)$, $(3, 1)$. (6) $(-4, 5)$, $(3, -2)$, $(0, -4)$.

2. Find the areas of the quadrilateral with vertices at

(1) $(0, 0)$, $(4, 6)$, $(2, 5)$, $(5, 1)$.
(2) $(2, -5)$, $(4, -1)$, $(-4, -2)$, $(1, 1)$.
(3) $(1, -2)$, $(2, 1)$, $(4, 2)$, $(3, -1)$.
(4) $(3, 3)$, $(1, 4)$, $(-2, 1)$, $(2, -3)$.

3. Show that the area of the quadrilateral whose vertices taken in order are $(x_1\ y_1)$, $(x_2\ y_2)$, $(x_3\ y_3)$, $(x_4\ y_4)$ is

$$\tfrac{1}{2}(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1 - x_1y_4 - x_2y_1 - x_3y_2 - x_4y_3).$$

4. Find the area of the triangle formed by $(-3,1)$, $(-1,5)$, $(1, 9)$. What inference can you draw from the result ?

5. Show that the following sets of points are collinear :
(1) $(-7, 1)$, $(-1, 4)$, $(3, 6)$. (2) $(-3, -7)$, $(-1, -4)$, $(5, 5)$.
(3) $(0, 2)$ $(4, 8)$, $(-8, -10)$. (4) $(5, 7)$, $(-3, 1)$, $(-7, -2)$

6. The points $(3, 4)$, $(-7, -6)$, $(9, y)$ are collinear; find y

7. Find the condition that the point (x, y) may be collinear with the points (1) $(-1, 1)$, $(2, 7)$, (2) $(1, 1)$, $(2, 3)$.

8. P, Q, R are the points $(3,1)$, $(5, -1)$, (x,y), find the equation to the locus of (x, y) given that the area of $\triangle PQR$ is $+5$.

9. A B are the points (2, 1), (5, 4); find the locus of the point $P(x, y)$ which is such that the area of $\triangle ABP$ is 6.

10. (x_1, y_1), (x_2, y_2) are two fixed points and (x, y) a variable point; write down the expression for the area of the triangle formed by the three points. Hence deduce the equation to the st. line passing through (x_1, y_1), (x_2, y_2).

11. A, B are two points on the x- and y-axes respectively such that $OA=a$ and $OB=b$, O being the origin. Find the equation to the locus of the point P which is such that the area of the quadrilateral OAPB is a constant $=k$.

12. The vertices of a quadrilateral, in order, are $(1, -4)$, $(4, 2)$, $(-2, 1)$, (x, y); if its area is $+28$, show that the locus of (x, y) has equation $5x+3y+30=0$.

13. Find the ratio in which the join of $(7, 3)$, $(-3, -2)$ divides the join of $(-10, 2)$, $(5, -3)$.

14. P, Q, R, S are the points $(-1, -2)$, $(2, -2)$, $(5, 8)$, $(-1, 3)$; find the ratio in which QS divides PR.

15. $P(2, 6)$, $Q(3, -2)$, $R(-3, -4)$, $S(-2, 13)$ are the vertices of a quadrilateral taken in order; find the ratios in which the diagonals PR, QS divide each other.

16. Find the ratios in which the diagonals of the quadrilateral $(8, 1)$, $(-6, 3)$, $(-7, 4)$, $(6, -3)$ divide each other. Find also the co-ordinates of the point of intersection.

17. $P(-15, -1)$, $Q(-1, -15)$, $R(13, 13)$, $S(-9, 9)$ are the vertices of a quadrilateral; find the ratios in which the diagonals PR, QS divide each other.

18 Using the result that the area of a triangle is also obtained as half the product of the base and the altitude, show that the length of the perpeneicular from the origin on the line joining (x_1, y_1), (x_2, y_2) is

$$(x_1y_2-x_2y_1)/\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}.$$

CHAPTER VII

The Straight Line

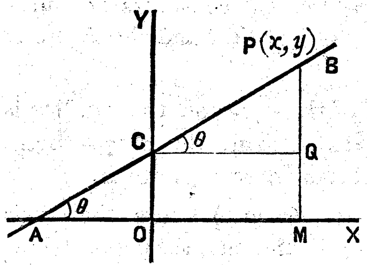
41. Equation to a Straight line.

We have already seen how the equation to a locus may be found when we are given the condition satisfied by the points on the locus and in many cases we have actually determined the equations to straight lines, circles and other curves. In particular, we have noticed in Art. 10, that the equation to a straight line || to the x-axis is of the form $y=k$, that of the x-axis itself being $y=0$. Similarly the equation to a straight line || to the y-axis is of the form $x=k'$ and that of the y-axis itself is $x=0$.

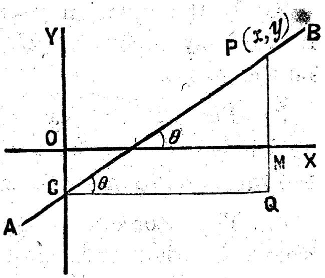
In this chapter and in the next, we derive the equation to st. line in various standard forms.

42. Gradient Form of the Equation to a Straight Line

Let AB be a st. line inclined at θ to the x-axis and cutting the y-axis in C, where $OC=c$. (Fig. 34 represents the case where c is positive while Fig. 35 represents the case where c is negative, θ being acute in both cases. The student should draw figures for the cases wherein θ is obtuse.)



(Fig. 34)



(Fig. 35)

Let P(x, y) be any point on the line AB and let CQ be drawn || to the x-axis to cut the ordinate of P in Q. Denoting the slope of AB by m, we have

$$m = \tan\theta = \frac{QP}{CQ} = \frac{MP - MQ}{CQ} = \frac{y - c}{x},$$

$$\therefore\ y - c = mx \quad \text{or} \quad y = mx + c.$$

which is the equation to the line AB.

43. Slope Form : Alternative Method. The slope of the join of P(x, y) and C$(0, c)$ is by Art. 28. $(y - c)/x$ and as long as P lies on the line AB this must be constant being always equal to m.

Hence, for *all* positions of P on AB, $(y - c)/x = m$, so that the equation to the st. line AB is $y = mx + c$.

Corollary. If the line AB passes through the origin, the constant $c = 0$ and the equation to AB then becomes $y = mx$.

Note 1. The segment OC of the y-axis, bounded by the origin and the line AB, is called the **intercept** made by the line on the y-axis.

2. In the equation $y = mx + c$, the constants m and c respectively denote the *slope* or *gradient* and the *intercept* on the y-axis.

3. The form $y = mx + c$ of the equation to a st. line is variously called as the **gradient form**, the **slope form**, the **tangent form** or the **slope-intercept form.**

4. The intercept is positive when AB cuts the y-axis above the origin and negative when it cuts the y-axis below.

44. Linear Equation represents a straight Line. It is to be observed that in the equation $y = mx + c$, the variables x, y occure in first degree only, that is, the equation is of the first degree in x and y, or as it is sometimes called a *linear* equation.

The converse is also true, i.e. a linear equation in x and y represents a st. line.

To prove this, consider the equrtion

$$px + qy + r = 0, \quad \ldots\ldots\ldots\ldots\ldots (1)$$

where p, q, r are any constants, and let (x_1, y_1) be a fixed point on the locus, so that $px_1 + qy_1 + r = 0.$ ....(2)
If (x_2, y_2) be any other point chosen at random any where on the locus, we have $px_2 + qy_2 + r = 0.$ ....(3)

Subtracting (3) and (2).

$$p(x_1 - x_2) + q(y_1 - y_2) = 0$$

or

$$(y_1 - y_2)/(x_1 - x_2) = -p/q,$$

which shows that the slope of the line joining (x_2, y_2) to the fixed point (x_1, y_1) is constant, being always equal to $-p/q$ hence (x_2, y_2) must lie on a st. line through (x_1, y_1) and of slope $-p/q$. That is, points satisfying equation (1) lie on a st. line.

45. Aliter. Take any two points (x_1, y_1), (x_2, y_2) on the locus so that we have

$$px_1 + qy_1 + r = 0,$$

$$px_2 + qy_2 + r = 0.$$

Multiplying the first of these by l, the second by k, (k and l being any two numbers arbitrarily chosen) and adding, we obtain $p(kx_2 + lx_1) + q(ky_2 + ky_1) + r(k + l) = 0.$
Divide throughout by $k + l$:

$$p\,\frac{kx_2 + lx_1}{k + l} + q\,\frac{ky_2 + ly_1}{k + l} + r = 0,$$

which shows that the point $\left(\dfrac{kx_2 + lx_1}{k + l},\ \dfrac{ky_2 + ly_1}{k + l}\right)$ lies on the locus $px + qy + r = 0.$
But this point is the point of devision of the line joining (x_1, y_1), (x_2, y_2) in the ratio $k : l$, where k and l are any

quantities chosen at pleasure ($k+l\neq0$). Hence the point dividing in any ratio the join of the two fixed points (x_1, y_1) (x_2, y_2) lies on the locus; but all such points of division lie on the line through (x_1, y_1), (x_2, y_2), so that it follows that the given equation represents a straight line.

Here is another alternative method:

Let (x_1,y_1), (x_2,y_2), (x_3,y_3) be any three points on the locus so that $px_1+qy_1+r=0$, $px_2+qy_2+r=0$, $px_3+qy_3+r=0$.

From the first two conditions we have

$$p:(y_1-y_2)=-q:(x_1-x_2)=r:(x_1y_2-x_2y_1).$$

But $p.q.r$ satisfy the third condition also;

$$\therefore\ x_3(y_1-y_2)-y_3(x_1-x_2)+(x_1y_2-x_2y_1)=0.$$

That is, the area of the triangle formed by the three points is zero; hence they are collinear.

Note 1. The form $px+qy+r=0$ of the equation of a line is called general form.

2. If we denote the expression $px+qy+r$ by L, the equation to the line may be written as $L=0$

46. Reduction to Slope Form. Given the equation to a st. line in the general form, we can easily convert it or *throw it* into the slope form. Thus, the equation $Ax+By+C=0$ can be written as

$$By=-Ax-C, \quad \text{or} \quad y=(-A/B)x+(-C/B),$$

which is the slope form required.

Also, we note that the slope of the line $Ax+By+C=0$ is $-A/B$, and the intercept of the line on the y-axis is $-C/B$,

Corollary 1. If $A=0$, the slope is 0; hence the line is $\parallel$ to the x-axis; while if $B=0$, the slope is ∞ and the equation reduces to $x=-C/A$, which is the equation to a line $\perp$ to the x-axis.

2. The condition that the two equations $Ax+By+C=0$ and $A'x+B'y+C=0$ may represent the same line is that the slopes, as well as the intercepts on the y-axis, in the two cases, should be equal; that is,

$$-\frac{A}{B}=-\frac{A'}{B'}, \qquad -\frac{C}{B}=-\frac{C'}{B'};$$

or

$$A/A'=B/B'=C/C';$$

i.e. the co-efficient of like terms should be proportional, and *not*, as the student might be tempted to say, necessarily equal

Example 1. *Find the equation to a line inclined at* 30° *to the x-axis and making an intercept* 3 *on the y-axis.*

The inclination of the line is 30°;

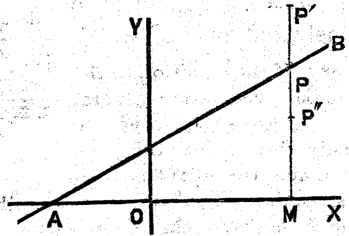
$$\therefore \text{ its slope} = \tan 30° = 1/\sqrt{3}.$$

The intercept of the line on the y-axis being 3, the required equation is $y=(1/\sqrt{3})x+3$, or $\sqrt{3}y=x+3\sqrt{3}$.

Example 2. *A st. line has equation* $4x-3y+15=0$; *find its slope, inclination to the x-axis and the intercept on the y-axis.*

The equation can be written as $3y=4x-15$, or $y=\frac{4}{3}x-5$. The intercept on the y-axis is -5; the slope of the line is $\frac{4}{3}$ the inclination of the line to the x-axis is the angle whose tangent is $\frac{4}{3}$, i.e. the inclination $=53°\ 8'$, (from table).

47. Positive and Negative Sides of a Line. If $P(h, k)$ be any points on the line $y=mx+c$, we have $y=mh+c$. Take two points $P'(h, k')$ and $P''(h, k'')$ on MP, the ordinate of P, so as to lie respectively above[1] and below the line (Fig, 36).



(Fig. 36)

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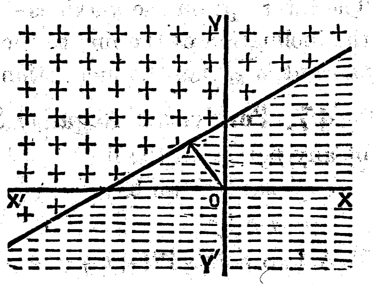
[1] By 'above' is ment that side in which y increases.

It is then readily seen that $k' > k > k''$,
or $k' > mh + c$, $k'' < mh + c$, since $k = mh + c$,
i.e. $k' - (mh + c) > 0$, $k'' - (mh + c) < 0$.

That is to say, for points above the line AB, the *expression* $y - (mx + c)$ becomes positive when the co-ordinates of the points are substituted for x, y, while, for points below the line AB, the *expression* $y - (mx + c)$ becomes negative when so substituted. In other words, if L stand for the expression $y - (mx + c)$, on one side of the line AB lie points whose co-ordinates, when substituted in L render it positive, while on the other side of AB lie points whose co-ordinates, when so substituted render L negative. The former side, on this account, is called the **positive side** of the line AB, while the later side is called the **negative side** of the line.

Thus, it will be observed, that the whole plane is divided into two distinct regions or domains, the positive domain and the negative domain, the former consisting of all those points whose co-ordinates make L positive and the latter consisting of all those points whose co-ordinates make L negative; and the boundary line between these two domains is the line L=0. This may be illustrated, as in Fig. 37.



(Fig. 35)

48. If the equation to the line be given in the general form $ax + by + c = 0$, it can be easily thrown into the slope form and the above reasoning applied. The student can easily varify that that side of the line is positive, the co-ordiaates of points on which make the expression $ax + by + c$ positive; similarly for the negative side. It is to be observed that if b and c are of like signs, the origin will be on the positive side of the line.

Note. While determining the positive and negative sides of a line, the equation to the line is to be so expressed as to make the co-efficient of y positive. In the case of a line parallel to the y-axis there is no y-term in the equation; for such a line, the right side or the side in which x is increasing, is defined as the positive side

49. Position of Points with respect to a line.

If A (x_1, y_1), B (x_2, y_2), be any two given points and $ax + by + c = 0$ any given line, it is readily seen that the points A and B are on the same side or on opposite sides of the line according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same sign or of opposite signs.

This result may also be deduced in the following manner.

Let the join of (x_1, y_1), (x_2, y_2) be cut by the line $ax + by + c = 0$ in the ratio $k : l$.

The co-ordinates of the point of section are

$$\frac{kx_2 + lx_1}{k + l}, \quad \frac{ky_2 + ly_1}{k + l},$$

Since this point lies on the given line, we have

$$a\,\frac{kx_2 + lx_1}{k + l} + b\,\frac{ky_2 + ly_1}{k + l} + c = 0;$$

multiplying throughout by $(k + l)$ and re-arranging the terms

$$k(ax_2 + by_2 + c) + l(ax_1 + by_1 + c) = 0,$$

from which $k/l = -(ax_1 + by_1 + c)/(ax_2 + by_2 + c)$.

But it is clear that the two points are on the same side or on opposite sides of the line according as $k : l$ is negative or positive, that is according as the expressions $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of like or unlike signs.

50. Alternatine Method. Let A (h, k), B (h', k') be two points on the line $ax + by + c = 0$, and let P(x_1, y_1) Q(x_2, y_2) be any two points that do not lie on the line.

Then $ah + bk + c = 0$, ...(1)

$ah' + bk' + c = 0$, ...(2)

$$\triangle PAB = \tfrac{1}{2}[x_1(k-k') - y_1(h-h') + (hk' - h'k)], \quad \ldots\ldots(3)$$
$$\triangle QAB = \tfrac{1}{2}[x_2(k-k') - y_2(h-h') + (hk' - h'k)]. \quad \ldots\ldots(4)$$

From (1) and (2) we have

$$\frac{a}{k-k'} = \frac{-b}{h-h'} = \frac{c}{hk'-h'k} = \frac{1}{\lambda}, \text{ say};$$
$$\therefore\ k-k' = \lambda a, \quad -(h-h') = \lambda b, \quad hk' - h'k = \lambda c.$$

Substituting these values in (3), (4), we obtain

$$\triangle PAB = \tfrac{1}{2}\lambda(ax_1+by_1+c), \quad \triangle QAB = \tfrac{1}{2}\lambda(ax_2+by_2+c);$$
$$\therefore\ \frac{\triangle PAB}{\triangle QAB} = \frac{ax_1+by_1+c}{ax_2+by_2+c}.$$

But P and Q are on the same side or on opposite sides of the line according as the area of the triangles PAB, QAB are of the same sign or of opposite signs. Therefore, it follows that P and Q are on the same side or on opposite sides of the line $ax+by+c=0$ according as the expressions ax_1+by_1+c and ax_2+by_2+c are of like or unlike signs.

51. Line through a Point with given Slope.

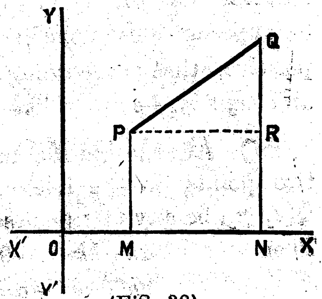
Let PQ be a line of slope m and passing through $P(x_1, y_1)$ and let $Q(x, y)$ be any point of the line. (Fig. 38).

The slope of the line through (x_1, y_1), (x, y) is, by Art. 28, Cor. 1, $(y-y_1)/(x-x_1)$; but the slope of the line PQ is given as m.

$$\therefore\ (y-y_1)/(x-x_1) = m,$$

or $\quad (y-y_1) = m(x-x_1).$

This is the condition satisfied by the co-ordinates of any point Q, and is, therefore, the equation to the line PQ.



(Fig. 38)

Another method of deriving the equation is this:

Let the equation to PQ be $y=mx+c$, the slope being given as m; since the line passes through (x_1, y_1), we have

$$y_1 = mx_1 + c,$$

Subtracting, $\quad y - y_1 = m(x-x_1),$

which is the equation required

Note. This form of the equation to a line is called the **point-slope form.**

52. Line through two Points.

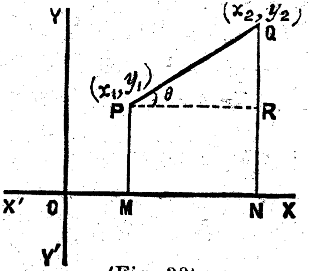
Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points and let PQ be inclined to the x-axis at an angle θ, Fig. 39.

Then, slope of the line $PQ = m = \tan\theta = \dfrac{RQ}{PR} = \dfrac{y_2-y_1}{x_2-x_1}$; (the slope can be obtained, even without a figure, by the application of Art. 28). Therefore, the equation to the line passing through (x_1, y_1) and having the above slope is, by Art. 51,

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

or
$$\frac{y-y_1}{x-x_1} = \frac{y_1-y_2}{x_1-x_2}$$

both being, of course, identical.



(Fig. 39)

Note. This form of the equation to a st. line is called the **two-point form.** As a corollary, it follows that the equation to the line through the origin and the point (x_1, y_1) is

$$y/x = y_1/x_1.$$

53. Alternative Method.

The result of the preceding Article may also be established in the following manner:

The slope of the line joining the two given points is $\dfrac{y_2-y_1}{x_2-x_1}$, if (x, y) be any point on the line, the slope is $\frac{y-y_1}{x-x_1}$.

But the slope of the line is the same by whichever method obtained;

$$\therefore \frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1},$$

which is the equation required. (*Vide* Art. 28, Cor. 2)

Another method is to assume the equation to the st. line through the points as

$$y=mx+c. \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

This passes through (x_1, y_1), (x_2, y_2); therefore

$$y_1=mx_1+c. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$y_2=mx_2+c. \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Subtracting (2) from (1), and (3) from (2), we obtain

$$y-y_1=m(x-x_1) \text{ and } y_1-y_2=m(x_1-x_2);$$

$$\therefore (y-y_1)/(x-x_1)=(y_1-y_2)/(x_1-x_2).$$

Example 1. *Find the equation of a line through* (1, 2) *and gradient* $\frac{1}{2}$.

The equation is $(y-2)/(x-1)=\frac{1}{2}$,

i.e. $x-2y+3=0$.

Example 2. *Find the equation of the line joining the points* $(-1, 3)$, $(2, -2)$, *and find the points at which the line cuts the co-ordinate axes.*

The equation to the line is $\frac{y-3}{x+1}=\frac{3+2}{-1-2}=-\frac{5}{3}$,

i.e. $3(x-3)+5(x+1)=0$, or $5x+3y-4=0$;

To find where the line cuts the x-axis, put $y=0$;

we have $5x-4=0$ or $x=\frac{4}{5}$;

that is, the line cuts the x-axis at the point $(\frac{4}{5}, 0)$

Similarly, by putting $x=0$, we find that the line cuts the y-sxis at the point $(0, \frac{4}{3})$.

Example 3. *Find the equation to the line through* $(-2, 3)$ *making intercept* 7 *on the y-axis.*

Since the intercept on the y-axis is 7, the equation is of the form $y=mx+7$; but the line passes through $(-2, 3)$; therefore $3=-2m+7$ which gives $m=2$.

Hence the equation to the line is $y=2y+7$.

Example 4. A$(-1, 6)$, B$(5, 2)$, C$(-3, 4)$ *are vertices of a triangle; find the equation to the median through* A.

D, the mid-point of the join of B, C is

$$\left(\frac{5-3}{2}, \frac{2+4}{2}\right) \quad \text{or} \quad (1, 3).$$

$$\therefore \text{ The equation to AD is } \frac{-6}{x+1}=\frac{6-3}{-1-1}=-\frac{3}{2},$$

or $2y-12=-3x-3$,

i.e. $3x+2y-9=0$.

That is, the equation to the median through A is

$$3x+2y-9=0.$$

Exercise XIII

1. Find the equation to the straight line having

(1) gradient 2, intercept on the y-axis 3.

(2) gradient $\frac{3}{4}$ intercept on the y-axis -2.

(3) intercept on the y-axis -4, inclined to the x-axis at 60°

(4) inclination to the x-axis 135°, passing through (0, 0).

(5) gradient -3 and passing through the origin.

(6) intercept on the y-axis 5 and inclined to the x-axis at an angle of $\tan^{-1}\frac{5}{12}$.

2. Throw the following lines, where necessary, into the gradient form and find the slope and the intercept on the y-axis in each case:

(1) $y=4x+7$ (2) $y=3x-9$ (3) $y+5x=11$

(4) $2y+7x+10=0$ (5) $ax+by=c$ (6) $x=ly+m$.

3. Find the condition that the following pairs of equations may represent identical lines ;

(1) $2x+3y+c=0$, $2x+3y+5=0$.
(2) $ax+by+c=0$, $lx+my+n=0$.
(3) $x/a+y/b=1$, $y=mx+c$.
(4) $y=mx+c$, $px+qy+r=0$.
(5) $x\cos\alpha+y\sin\alpha=p$, $ax+by+c=0$.
(6) $lx+my+n=0$, $3x+4y+6=0$.

4. Find the equations to the following straight lines :

(1) of gradient 3 and passing through the origin.

(2) of gradient $-\frac{4}{3}$ and passing through the point (2, 1).

(3) of gradient 0 and passing through (3, − 5).

(4) of inclination 45° to the x-axis and passing through the point (1, 5).

(5) inclined to the x-axis at 120° and passing through (− 4, 7).

(6) inclined at 150° to the x-axis and passing through $(2\sqrt{3}, -1)$.

(7) passing through (− 1, 4) and making intercept − 3 on the y-axis.

(8) passing through (3, 6) and making intercept − 15 on the y-axis.

5. Find the equations to the st. lines joining the following pairs of points.

(1) (0, 0), (3, 4) (2) (− 2, 4), (1, − 2) (3) (1, 2), (3, 4)
(4) (− 1, −3), (−5, − 7) (5) $\frac{3}{2}, -\frac{1}{2}), (\frac{5}{2}, -\frac{7}{2})$
(6) $(-\frac{1}{3}, \frac{3}{4}), (-\frac{2}{3}, \frac{1}{4})$ (7) $(a, b), (b, a)$
(8) $(at_1^2, 2at_1), (at_2^2, 2at_2)$
(9) $(a\cos\theta, a\sin\theta), (a\cos\phi, a\sin\phi)$
(10) $(a\cos\theta_1, b\sin\theta_1), (a\cos\theta_2, b\sin\theta_2)$

6. Find the equations to the sides of the triangle formed by

(1) (1, 2), (4, 3), (3, − 1) (2) (2, 5), (3, − 3), (− 1, − 4).

7. Find the equation to the line through (4 , − 1) and bisecting the join of (1, −2) and (− 3, 3), and find the points at which the line meets the axes.

8. Find the equations to the medians of the triangle whose vertices are

(1) (3, 1), (1, −2), (− 2, 4) (2) (4, 3), (− 8, 5), (6, − 1)

9. Find the equations to the sides and the medians of the triangle whose vertices are (1, − 5), (2, 7), (6, 3).

54. Parallel Lines. We have already seen, in Art. 26. Cor., that the slopes of parallel lines are equal ; and conversely. Hence, it follows that the lines

$$y=mx+c, \qquad y=mx+c'$$

are parallel to each other and both are parallel to $y=mx$, which is a line through the origin.

If the equations of the two lines are given in the form

$$ax+by+c=0, \quad a'x+b'y+c'=0,$$

the condition that they may be parallel is obtained by equating their slopes;

i.e, $-a/b=-a'/b'$ or $a/b=a'/b'$.

This condition may also be expressed as

$$a/a'=b/b' \quad \text{or as} \quad ab'-a'b=0.$$

Cor. The equation $ax+by=0$ represents a line through the origin, parallel to $ax+by+c=0$.

55. Equation to a Parallel Line. It is easily varified that the equation to a line parallel to $ax+by+c=0$ is of the form $ax+by+k=0$, (k being any constant), since both of them are parallel to $ax+by=0$.

If it be given, in addition, that the parallel line passes through the point (x_1, y_1), the value of k can be computed and the equation to the parallel determined completely; for, expressing analytically the condition that the parallel passes through (x_1, y_1), we get

$$ax_1+by_1+k=0.$$

which gives $\qquad k=-(ax_1+by_1).$

The equation to the parallel line is thus

$$ax+by=ax_1+by_1 \quad \text{or} \quad a(x-x_1)+b(y-y_1)=0.$$

Alternatively, the equation may be derived by making use of the point-slope form of the equation to a st. line. For, the parallel line has the slope $-a/b$ and passes through (x_1, y_1); its equation, therefore, is $(y-y_1)/(x-x_1)=-a/b$,

or $$a(x-x_1)+b(y-y_1)=0.$$

Example 1. *Find the equation to a st. line through* (1, 1) *parallel to the line* $2x+3y=7$.

The equation to a parallel line being $2x+3y=k$, the condition that this should pass through (1, 1) is that

$$2.1+3.1=k, \qquad \text{whence} \qquad k=5.$$

∴ The equation to the line through (1, 1) ∥ to $2x+3y=7$ is

$$2x+3y=5.$$

Example 2. *Show that the join of* $(1, -2)$, $(-1, 3)$ *is* ∥ *to the line* $5x+2y-7=0$.

The slope of the line joining the given points is

$$\frac{3+2}{-1-1}=-\frac{5}{2}.$$

Throwing the given equation into the slope form, we get

$$y=-\tfrac{5}{2}x+\tfrac{7}{2},$$

which shows that the slope of the given line is $-\frac{5}{2}$.

Hence, the line joining $(1, -2)$, $(-1, 3)$ and the line $5x+2y-7=0$ have the same slope; they are, therefore parallel.

Example 3. *Find tke equation to the line through* $(-2, 3)$ *parallel to the join of* $(2, -5)$, *and* (4, 1).

The gradint of the join of the two given points is

$$\frac{1+5}{4-2}=\frac{6}{2}=3.$$

∴ The equation to a line through $(-2, 3)$ and having slope 3 is $\quad (y-3)=3(x+2) \quad$ or $\quad 3x-y+9=0.$

56. Perpendicular Line. It has been shown, in Art. 30, that if two st. lines are at right angles, the product of their slopes is -1, and conversely. Hence, the two lines $y=mx+c$ and $y=m'x+c'$ are at right angles if

$$mm'=-1, \quad \text{or} \quad m'=-1/m.$$

[Note that c, c', intercepts on the y-axis, do not enter into the condition for perpendicularity.]

So if a line has equation $y=mx+c$, any line perpendicular to this will be of the form $y=(-1/m)x+k$, where k is any constant.

If the equations to the lines be given as

$$ax+by+c=0, \quad a'x+b'y+c'=0,$$

then the condition for perpendicularity is readily seen to be

$$(-a/b)\times(-a'/b')=-1.$$

i.e. $$aa'=-bb', \quad \text{or} \quad aa'+bb'=0.$$

Note. The slope of a line ⊥ to $ax+by+c=0$ is b/a.

57. Equation to a Perpendicular Line. The student can easily varify that the equation to any line perpendicular to $ax+by+c=0$ is $bx-ay=k$, where k may be any constant.

If any other condition regarding the perpeneicular be given, the constant k can be evaluated and the equation to the perpendicular completely determined.

Thus, if the condition be that the perpendicular passes through a given point (x_1, y_1), then expressing this condition analytically, we have $\; bx_1-ay_1=k.$

which gives the value of k; hence, the equation to the perpendicular is

$$bx - ay = bx_1 - ay_1 \quad \text{or} \quad b(x - x_1) = a(y - y_1).$$

This equation may also be derived by writing the equation of line through (x_1, y_1) in the form $y - y_1 = m(x - x_1)$, where m is the slope.

If the line is to be $\perp$ to $ax + by + c = 0$ its slope is b/a. Hence, the equation to the perpendicular is

$$(y - y_1) = (x - x_1)\,b/a \quad \text{or} \quad b(x - x_1) = a(y - y_1).$$

Example 1. *Find which of the following lines are at right angles:*

$$2x + 3y + 6 = 0, \quad 3y = 2y + 1, \quad 4x + 6y = 15.$$

When thrown into the slope form these equations become

$$y = -\tfrac{2}{3}x - 2, \quad y = \tfrac{3}{2}x - \tfrac{1}{2}, \quad y = -\tfrac{2}{3}x + \tfrac{5}{2},$$

thus showing that the first and the third lines are parallel and that both of these are perpendicular to the second line.

Example 2. *Find the equation to a line through* (1, 3) *perpendicular to* (1) $y = 2x + 9$. (2) $3x - 5y + 4 = 0$.

(1) The slope of the first line being 2, that of the $\perp$ is $-\tfrac{1}{2}$; and since the $\perp$ passes through (1, 3) its equation is

$$(y - 3) = -\tfrac{1}{2}(x - 1) \quad \text{or} \quad x + 2y - 7 = 0.$$

(2) Any line $\perp$ to $3x - 5y + 4 = 0$ is of the form

$$5x + 3y = k;$$

if this passes through (1, 3), we have

$$5.1 + 3.3 = k, \quad \text{or} \quad k = 14.$$

Hence, the equation to the line $\perp$ to $3x - 5y + 4 = 0$ and passing through (1, 3) is $5x + 3y = 14$.

Example 3. *Find the equation to a line perpendicular to* $3x + 4y + 1 = 0$ *and making an intercept* $-\tfrac{2}{3}$ *on the y-axis.*

The equation to the $\perp$ is of the form $4x - 3y = k$.

Since the $\perp$ makes intercept $-\tfrac{2}{3}$ on the y-axis, the point $(0, -\tfrac{2}{3})$ lies on the $\perp$; this gives the value of k as

$$4.0 - 3.(-\tfrac{2}{3}) \quad \text{or} \quad 2.$$

The equation to the perpendicular is, therefore, $4x - 3y = 2$.

Alternative Method. The slope of the given line is $-\tfrac{3}{4}$ and hence the slope of the perpendicular is $\tfrac{4}{3}$. Also the intercept of the perpendicular on the y-axis is $-\tfrac{2}{3}$,

$\therefore$ equation to the perpendicular is $y = \tfrac{4}{3}x - \tfrac{2}{3}$.

or $\quad 4y = 4x - 2$.

Example 4. *Find the equation to a line through* (4, 1) *and* $\perp$ *to the joining* (−1, 1) *and* (5, −2).

Slope of the join of (−1, 1) and (5, −2) is $\dfrac{1+2}{-1-5}$ or $-\tfrac{1}{2}$,

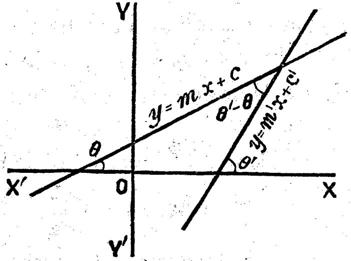
$\therefore$ slope of the $\perp$ is 2.

The equation to the line through (4, 1) and of slope 2 is

$$y - 1 = 2(x - 4) \quad \text{or} \quad y = 2x - 7.$$

58. Angle between two Lines.

Let $y = mx + c$, $y = m'x + c'$ be the equations to two given st. lines, inclined to the x-axis at angles θ, θ' respectively. Then $\tan\theta = m$, tad $\theta' = m'$.



(Fig. 40)

The angle between the two lines is (Fig. 40) $(\theta' - \theta)$. [This result could be obtained even without the aid of a figure; for, the two given lines are respectively parallel to $y = mx$, $y = m'x$. both of which passes through the origin, the first inclined at θ to the x-axis, the second at θ'. The angle between the latter pair of lines, namely, $\theta' - \theta$, is equal to the angle between their parallels $y = mx + c$ and $y = m'x + c'$.

Knowing m, m' we can find θ, θ' (by referring to Trigonometrical Tables, if necessary), and hence we may find $(\theta' - \theta)$, the angle between the two given lines.

Alternatively, we use the relation

$$\tan(\theta' - \theta) = \frac{\tan\theta' - \tan\theta}{1 + \tan\theta' \tan\theta} = \frac{m' - m}{1 + mm'},$$

from which we can find $(\theta' - \theta)$, since m, m' are known.

If we denote the angle between the lines ϕ, then

$$\tan\theta = \tan(\theta' - \theta) = \frac{m' - m}{1 + mm'},$$

or

$$\phi = \tan^{-1}\frac{m' - m}{1 + mm'},$$

a symbol which means that ϕ is such an angle whose tangent is $(m' - m)/(1 + mm')$.

Corollary. As an immediate corollary, we find that when $m - m'$, $\phi = 0$, that is, the lines are parallel; and when $mm' = -1$, $\tan\phi = \infty$, so that $\phi = 90°$, that is, the lines are at right angles. These results have already been established by more direct methods.

59. Angle between two lines : General Form.

If the equations to the two lines be given in the general form,

$$ax + by + c = 0, \quad a'x + b'y + c' = 0,$$

we know that the slopes of these lines are $-a/b$, $-a'/b'$;

therefore,
$$\tan\phi = \frac{-a'/b' - (-a/b)}{1 + (-a'/b')(-a/b)} = \frac{ab' - a'b}{aa' + bb'};$$

$$\therefore \phi = \tan^{-1}\frac{ab' - a'b}{aa' + bb'}.$$

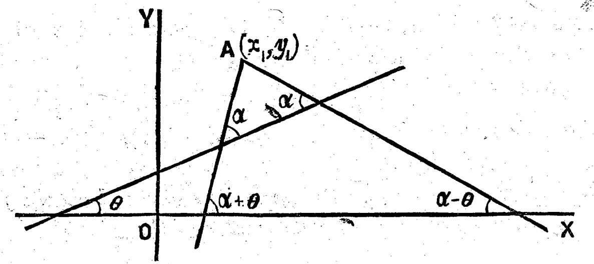
This result can be expressed in words thus:

$$\phi = \tan^{-1}\frac{\text{difference of cross products}}{\text{sum of direct production}}.$$

From this, as a corollary, it follows that the lines are parallel if $ab' - a'b = 0$ and perpendicular if $aa' + bb' = 0$, results which have already been deduced in Arts 54, 56.

60. Lines inclined at a given Angle to a given Line.

Let $y = mx + c$ be any line inclined to the x-axis at an angle θ; through a given point (x_1, y_1) it is possible to draw two lines, in general, making a given angle α with the line $y = mx + c$, as indicated in Fig. 41. The inclination of one of the lines to the x-axis is evidently $\theta + \alpha$ and that of the other is $180° - (\alpha - \theta)$.



(Fig. 41)

Their slopes are
$$\tan(\theta + \alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} = \frac{m + \tan\alpha}{1 - m\tan\alpha},$$

and
$$\tan\{180° - (\alpha - \theta)\} = \tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha},$$
$$= \frac{m - \tan\alpha}{1 + m\tan\alpha}.$$

Hence, the equations to the two lines throgh (x_1, y_1) and inclined to the line $y = mx + c$ at an angle α are,

$$\frac{y - y_1}{x - x_1} = \frac{m + \tan\alpha}{1 - m\tan\alpha} \quad \text{and} \quad \frac{y - y_1}{x - x_1} = \frac{m - \tan\alpha}{1 + m\tan\alpha}.$$

61. Intersection of two Lines.

The method of finding the points of intersection of any two loci has already been dealt with in Art 12. Following that method we see that, if (h, k) be the point of intersection of the two st. lines

$$a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0.$$

then (h, k) should satisfy both the equations, that is,

$$a_1h+b_1k+c_1=0, \quad a_2h+b_2k+c_2=0.$$

Solving these two equations for $(h. k)$, we have, by the usual *Rule of Cross-multiplication*,

$$h=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \quad k=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1},$$

which determine the point of intersection.

Corollary. If $a_1b_2-a_2b_1=0$, the co-ordinates of the point of intersection are infinite, that is, the lines are parallel, and conversely. This result agrees with that of Art. 54.

Example 1, *The points* A(2, 5), B(−2, 1), C(4, −1) *are the vertices of a triangle. Find the equations to the perpendiculars from* A *and* B *on the opposite sides and obtain from them the co-ordinates of the ortho-centre of the triangle.*

The slope of the side BC is $\dfrac{1+1}{-2-4}=-\frac{1}{3}$;

∴ slope of ⊥ from A on BC is 3, so that the equation to the ⊥ is $(y-5)=3(x-2)$

or $3x-y-1=0.$ ....(1)

Similarly, the slope of the side CA is $\dfrac{5+1}{2-4}$ or -3;

∴ slope of the ⊥ from B on CA is $\frac{1}{3}$, so that the equation to the perpendicular is $(y-1)=\frac{1}{3}(x+2)$

or $x-3y+5=0.$ ....(2)

The point of intersection of (1) and (2) is the ortho-centre.

Solving (1) and (2), we have $x=1$, $y=2$.

∴ The co-ordinates of the ortho-centre are (1, 2).

Note. The centroid G of the triangle ABC is $(\frac{4}{3}, \frac{5}{3})$. Hence OG divided internally in the ratio of 2: 1 gives the Nine-point centre N, and divided externally in the same ratio gives the circum-centre S. We can thus obtain the co-ordinates of the Nine-point centre and the circum centre of the given triangle.

Example 2. *Find the angles of the triangle whose sides are* $x+2y+1=0$, $3x-y=0$, $2x+y+3=0$.

The tangents of the three angles of the triangle are ± 7, ± 1, $\pm\frac{3}{4}$.

If A is the greatrst angle (acute or obtuse), then B and C are both acute and

$$|\tan A| > |\tan B| \text{ or } |\tan C|.$$

Hence the two numerically smaller of the tangents fix B and C. *

Thus $\tan B=+\frac{3}{4}$, $\tan C=1$, so that

$$B=36°52'. \qquad C=45°.$$

$$\therefore A=180°-(36°52'+45°)=98°8'.$$

The angles of the triangle are therefore 98° 8′, 36°52′, 45°

Exercise XIV

1. Test the following pairs of lines for parallelism and perpendicularity, and when the lines are neither parallel nor perpendicular, find the angle between them :

(1) $2x+y+1=0$, $4x+2y+7=0$.
(2) $4x+3y+2=0$, $3x-4y-1=0$.
(3) $y=4x+3$, $4y+x=9$. (4) $x-3y+4=0$, $y+\frac{1}{2}=\frac{1}{3}x$.
(5) $7x+4y=7$, $4x+7y=5$. (6) $\frac{1}{2}y-x=1$, $2y+x-3=0$.
(7) $x+5y+7=0$, $10y=3-2x$. (8) $x=3y+1$, $x=5y$.
(9) $7x+5y+3=0$, $5x-7+3y=0$.

2. Show that the join of (1, 2), (3, 4) is parallel to the line $x-y+1=0$.

3. Show that the line joining (−1, 4), (2, −3) is parallel to the line $7x+3y=11$.

4. Show that the line joining (0, 5), (−1, 3) is perpendicular to the line $x+2y=3$,

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* The author is indedted to Mr. A. A. Krishnaswamy Iyengar for suggesting this method.

5. Show that the line joining $(-4, -5)$, $(2, -1)$ is perpendicular to the line $12x+8y+5=0$.

6. Find the angle between the join of $(-3, -2)$, $(5, 4)$ and the line $x-3y+7=0$.

7. Find the equation to the line through $(5, 2)$ and $\parallel$ to the x-axis.

8. Find the equation to the line $\parallel$ to $x=2y$ and passing through $(4, 3)$.

9. Obtaing tho equation to the line $\parallel$ to $5x+3y=7$ and passing through $(-1, -2)$.

10. Find the equation to the line through $(1, 7)$ parallel to the line $x+y=0$.

11. Find the equation of the line through $(-2, 5)$ parallel to the line $4x-7y=8$.

12. A st. line is drawn through $(3, 3)$ perpendicular to the line $3x-7y=4$; find its equation.

13. Find the equation of the line through $(2, 5)$ perpendicular to the line $4x+y=0$.

14. Find the equation of the line through $(-3, -1)$ and perpendicular to the line $5x-2y+3=0$.

15. Determine the equation of a line through $(\frac{1}{2}, -\frac{2}{3})$ and perpendicular to the line $6x+y=5$, and find the points at which the perpendicular meets the axes.

16. A perpendicular is drawn to the line $2x-9y+1=0$ so as to cut off an intercept 4 on the y-axis; find the equation to the perpendicular.

17. Find the equations to the st. lines through the origin inclined at 30° to the line jaining $(1, 3)$, $(5, 7)$.

18. Show that the equations to the st. lines passing through $(3, -2)$ and inclined at 60° to the line $\sqrt{3}x+y=1$ are $y+2=0$ and $y-\sqrt{3}x+2+3\sqrt{3}=0$.

19. Obtain the equations to the two st. lines which pass through the point $(2, 6)$ and make an angle $\pi/4$ with the st. line $y=2x+5$.

20. Calculate the tangents of the angles of the triangle whose sides are the lines $2x-3y-5=0$, $5x-y-3=0$, $x+y+5=0$. Also find the angles of the triangle.

21. Find the equation to a line through $(5, -4)$ and $\parallel$ to the join of $(1, -6)$, $(-3, 3)$.

22. Obtain the equation to the line through $(2, 2)$ and $\perp$ to the join of $(-1, -4)$, $(3, -2)$; also find the points at which the perpendicular cuts the co-ordinate axes.

23. Find the equation to the line through the point $(-4, 2)$ and $\perp$ to the line joinining $(10, 3)$, $(-8, -5)$.

24. A$(6, 4)$, B$(5, -3)$, C$(-1, 2)$ are the vertices of a triangle, find the equation to the altitude drawn from A.

25. Obtain the equations to the altitudes of the triangle whose vertices are $(1, -3)$, $(2, 5)$, $(-1, 4)$.

26. The points A$(2, 3)$, B$(5, 4)$, C$(3, -2)$ are the vertices of a triangle; find the equations to the perpendiculars from A and B on the opposite sides and obtain the co-ordinates of the ortho-centre.

27. Find the ortho-centre of the triangle whose vertices are $(-1, 4)$, $(1, 3)$, $(2, 7)$.

28. Show that the ortho-centre of the triangle formed by $(1, 5)$, $(-3, 1)$, $(3, -1)$ lies on the y-axis.

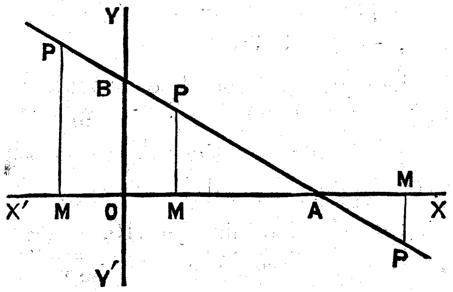
29. Obtain the co-ordinates of the ortho-centre of the triangle formed by the points $(-6, -8)$, $(12, 1)$, $(-6, 5)$.

30. Find the co-ordinates of the ortho-centre of the triangle whose vertices are $(5, 2)$, $(-1, 1)$, $(2, 7)$.

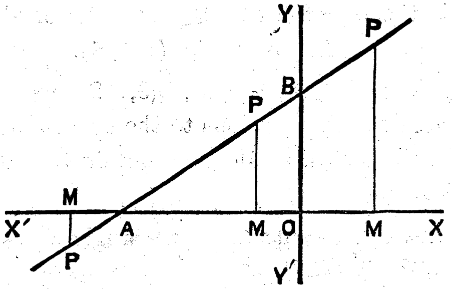
CHAPTER VIII

The Straiget Line - (Continued)

62. Intercept Form of the Equation to a Straight Line. Let AB be a straight line intersecting the co-ordinate axes at A, B making OA$=a$, OB$=b$, in sign and in magnitude. (Fig. 42, 43).



(Fig. 42)



(Fig. 43)

Take any point P(x, y) on the line AB and let MP be the ordinate of P so that OM$=x$, MP$=y$.

From the similar triangles AMP, AOB, we have

$$\frac{\text{MP}}{\text{OB}}=\frac{\text{MA}}{\text{OA}}.$$

This equality is true, considering both the sign and magnitude of the quantities involved, in respect of both the figures and for all possible positions of P. Further,

$$\text{OA}=a,\ \text{OB}=b,$$

$$\text{MP}=y,\ \text{MA}=\text{MO}+\text{OA}=-\text{OM}+\text{OA}=-x+a.$$

Substituting these values in the above equality, we obtain

$$\frac{y}{b}=\frac{a-x}{a}=1-\frac{x}{a};$$

so that $x/a+y/b=1,$

which is thus the equation to the line AB cutting off intercepts a and b on the co-ordinate axes.

The above form of the equation to a st. line is called the **intercept from.**

36. Alternative methods of deriving the Intercept Form. The result established in the preceding Article can be derived in a variety of other ways, some of which are indicated below.

1. The sum of the areas of the triangles OAP, OPB, taken with their proper sign, is equal to the area of the triangle OAB, so that

$$\tfrac{1}{2}a\cdot y+\tfrac{1}{2}b\cdot x=\tfrac{1}{2}a\cdot b,$$

whence $x/a+y/b=1.$

2. The line AB makes an intercept b on the y-axis and is inclined to the x-axis at an angle whose tangent is $-b/a$; hence, making use of the gradient from, the equation to the st. line AB is obtained as

$$y=-\frac{b}{a}x+b \quad \text{or} \quad \frac{x}{a}+\frac{y}{b}=1.$$

3. We may also make use of the two-point form of the equation to the line, since we know that $(a, 0)$, $(0, b)$ are points on the line; this gives the equation as

$$\frac{y-b}{x-0}=\frac{b-0}{0-a}, \quad \text{or} \quad a(y-b)+bx=0,$$

i. e. $x/a+y/b=1$.

4. If (x, y) be any point on the line AB, the area of the triangle formed by $(a, 0)$, $(0, b)$, (x, y) is 0, that is

$$\tfrac{1}{2}[a(b-y)+0\,(y-0)+x(0-b)]=0.$$

or $ab-ay-bx=0$, i.e. $x/a+y/b=1$.

64. Reduction to Intercept Form.

If we are to find the intercepts made by any line on the co-ordinate axes, it is only necessary to throw the given equation into the intercept from; we can then "read off", as it were, the intercepts.

Thus, if the given line be $ax+by+c=0$, the intercept form is

$$\frac{x}{(-c/a)}+\frac{y}{(-c/b)}=1,$$

whence the intercepts on the x- and y-axes are respectively $-c/a$ and $-c/b$.

Example 1 *Find the equation to the st. line making intercepts* -3 *and* $4\frac{1}{2}$ *on the co-ordinate axes,*

The equation is $$\frac{x}{-3}+\frac{y}{4\frac{1}{2}}=1,$$

which simplifies to $3x-2y+9=0$.

Example 2. *Find the intercepts on the co-ordienate axes made by the line* $2x-3y-4=0$

Throwing the given equation into the intercept form, we get

$$\frac{x}{2}+\frac{y}{-4/3}=1,$$

from which it is readily seen that the intercepts on the axes are 2 and $-\frac{4}{3}$.

Example 3. *Find the equation to the straight line which passes through the point* (3, 4) *and which is such that the portion of the line between the axes is divided by the point in the ratio* 2: 1.

Let the equation of the line be $x/a+y/b=1$. This meets the x-axis at $(a, 0)$ and the y-axis at $(0, b)$. The co-ordinates of the point dividing the line joining these two points in the ratio 2 : 1 are $(a/3, 2b/3)$.

If this coincides with the point (3, 4), we have

$$a/3=3, \quad 2b/3=4,$$

so that $$a=9, \quad b=6.$$

Hence the equation to the st. line is $x/9+y/6=1$.

Exercise XV

Find the equation to the st. line determined by the following data :

1. x-intercept $=5$, y-intercept $=2$.
2. x-intercept $=-2$, y-intercept $=3\frac{1}{3}$.
3. x-intercept $=-\frac{3}{4}$, y-intercept $=-\frac{1}{2}$.
4. x-intercept $=-6$, y.intercept $=2\sqrt{3}$.
5. x-intercept $=p\sec\alpha$, y-intercept $=p\operatorname{cosec}\alpha$.
6. Slope $=3$, x-inrercept $=2$.
7. Slope $=-\frac{5}{12}$, x-intercept $=-6$.

8. Passing through (2, 3) and making on the co-ordinate axes intercepts which are (i) equal in sign and in magnitude (ii) equal in magnitude but opposite in sign.

9. Passing through $(-1, -5)$ and having the x-intercept thrice the y-intercept.

10. Passing through $(-3, 4)$ and making on the co-ordinate axes intercepts in the ratio $4 : -7$.

11. Find the equation to a st. line which passes through the point A(3, 5) and is such that the part intercepted between the axes is bisected at A.

12. Find the equation to the st. line which passes through $(-3, -4)$ and is such that the portion of it between the co-ordinate axes is divided at the point in the ratio 3 : 5.

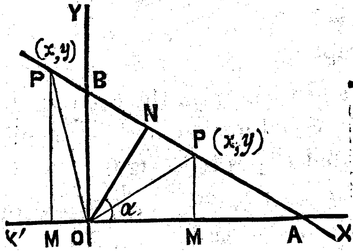
13. A straight line passes through (2, 5); when the intercepts on the axes are interchanged, the line passes through $(1, -\frac{2}{3})$; find the equation to the line.

14. Find the intercepts made on the co-ordinate axes by the line joining the points (x_1, y_1), (x_2, y_2).

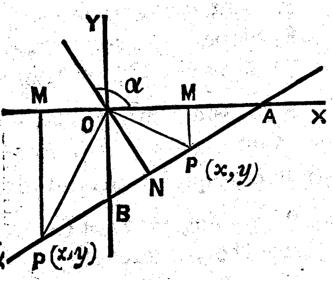
15. Two straight lines are such that the x-intercept of one equals the y-intercept of the other, in sign and magnitude, and vice versa; show that the locus of the intersection of the two lines $x - y = 0$.

65. Normal Form of the Equation to a Straight Line.

Let AB be any st. line cutting the co-ordinate axes at A and B and let ON be the ⊥ from the origin to the line; let $ON = p$, α=the angls which ON makes with the x-axis. Let $P(x, y)$ be any point on AB, and MP the ordinate of P. (Fig. 44, 45). It is readily seen that in the former figure the angle between the perpendicular and the y-axis is $90° - \alpha$, while in the later it is $\alpha - 90°$. Note that the cosines of these angles, however are equal.



(Fig. 44)



(Fig. 45)

Now, by Art. 5, the projection of OP on the line ON
=the sum of the projections of OM, MP on ON,

i.e. $$ON=OM \cos\alpha+MP \cos(90°-\alpha);$$

$$\therefore\ p=x \cos\alpha+y \cos(90°-\alpha.$$

or $$x \cos\alpha+y \sin\alpha=p,$$

which is the required equation to the line AB.

This form of the equation to a straight line is called the **perpendicular form** or the **normal form.**

66. Normal form : Alternative Methods.

Other methods of deriving the equation to a st. line in the perpendicular form are given below:

1. Draw MQ ⊥ to ON and PL ⊥ to MQ. Then, in Fig. 44, the angle $PML=\alpha$,

$$\therefore\ ON=OQ+QN=OQ+LP$$

i.e. $$p=x \cos\alpha+y \sin\alpha.$$

In Fig. 45, the angle $PML=180°-\alpha$, and remembering that ON is the negative direction of the perpendicular,

$$ON=OQ+QN=-OM \cos(180°-\alpha)+MP \sin(180°-\alpha)$$

i.e. $$p=x \cos\alpha+y \sin\alpha.$$

Hence, in both cases, we obtain the equation as

$$x \cos\alpha+y \sin\alpha=p,$$

2. The co-ordinates of N are $(p \cos\alpha, p \sin\alpha)$;

$\therefore$ the gradient of PN is $(y-p \sin\alpha)/(x-p \cos\alpha)$.

Also the gradient of ON is $\tan\alpha$.

But ON, PN are right angles; hence the product of their slope is -1; that is,

$$\frac{y-p \sin\alpha}{x-p \cos\alpha}\times\frac{\sin\alpha}{\cos\alpha}=-1,$$

which simplifies to $y \sin\alpha-p \sin^2\alpha=-x \cos\alpha+p \cos^2\alpha$

or $$x \cos\alpha+y \sin\alpha=p \cos^2\alpha+p \sin^2\alpha=p$$

3. From the right-angled triangle ONA, we have

$$OA=ON \sec AON=p \sec\alpha,\ \text{in Fig. 44,}$$

$$OA=-ON \sec(180°-\alpha)=p \sec\alpha,\ \text{in Fig. 45;}$$

similarly $OB=ON \sec NOB=p \sec(90°-\alpha)$

$$=p \operatorname{cosec}\alpha,\ \text{in Fig. 44,}$$

$$OB = ON \sec \angle NOB = p \sec (\alpha - 90^\circ)$$
$$= p \operatorname{cosec} \alpha, \text{ in Fig. 45.}$$

Hence, OA, OB being the intercepts made by the line AB on the co-ordinate axes, the equation to AB, in both the cases, is $x/(p \sec \alpha) + y/(p \operatorname{cosec} \alpha) = 1$,

or $x \cos \alpha + y \sin \alpha = p$.

4. Let $OA = r$ and $\angle AOP = \theta$; then $\angle PON = \alpha - \theta$, in Fig. 44. But $OP \cos \angle PON = ON$,

i.e. $r \cos (\alpha - \theta) = p$,

or $r \cos \alpha \cos \theta + r \sin \alpha \sin \theta = p$,

or $x \cos \alpha + y \sin \alpha = p$, since $x = r \cos \theta$, $y = r \sin \theta$.

With slight variations in the intermediate steps, the final result is true in the case of Fig. 45 also.

5. Referring to Fig. 44, the length OB is $p \operatorname{cosec} \alpha$ and $\angle BAX = 90^\circ + \alpha$, so that the slope of AB is $\tan (90^\circ + \alpha)$ or $-\cot \alpha$.

$\therefore$ The equation to the line AB, using the gradient form, is

$$y = -x \cot \alpha + p \operatorname{cosec} \alpha,$$

whence $x \cos \alpha + y \sin \alpha = p$.

The proof of this in the case of Fig 45 is left as an exercise for the student.

6. Considering the areas, with their proper signs, we have $\triangle OAB = \triangle OAP + \triangle OPB$,

i.e. $\frac{1}{2} OA . OB = \frac{1}{2} OA . MP + \frac{1}{2} OB . OM$,

or $p \sec \alpha . p \operatorname{cosec} \alpha = p \sec \alpha . y + p \operatorname{cosec} \alpha . x$;

multiplying throughout by $(\sin \alpha \cos \alpha)/p$,

$$x \cos \alpha + y \sin \alpha = p.$$

67. Reduction to Normal Form.

Given the equation to a st. line as $ax + by + c = 0$, it may be required to throw it into the normal form.

The normal form of the equation to a line being of the type $x \cos \alpha + y \sin \alpha - p = 0$.

we have to find the values of α and p which will make the two equations represent identical lines. The condition for this, by Art. 46, Cor. 2, is that

$$\frac{a}{\cos \alpha} = \frac{b}{\sin \alpha} = \frac{c}{-p} = r \text{ (say)},$$

so that $a = r \cos \alpha$, $b = r \sin \alpha$, $c = -pr$.

$$\therefore a^2 + b^2 = r^2(\cos^2 \alpha + \sin^2 \alpha) = r^2,$$

$$\therefore r = \sqrt{a^2 + b^2}, \text{ thus giving}$$

$$\cos \alpha = \frac{a}{\sqrt{(a^2+b^2)}}, \quad \sin \alpha = \frac{b}{\sqrt{(a^2+b^2)}}, \quad p = \frac{-c}{\sqrt{(a^2+b^2)}}.$$

Thus the normal form of the given equation $ax + by + c = 0$.

is $$\frac{a}{\sqrt{(a^2+b^2)}} x + \frac{b}{\sqrt{(a^2+b^2)}} y + \frac{c}{\sqrt{(a^2+b^2)}} = 0.$$

There is, of course, ambiguity of sign before the radical; but, since α lies between 0 and π the value of $\sin \alpha$ is always positive, so that the positive or negative sign is to be prefixed to the radical according as b is positive or negative.

Note. It is important to remember that when the equation to a st. line is expressed in the normal form, the sum of the squares of the co-efficients of x and y should always be equal to unity.

68. Rule for Reducing Equation to Normal Form.

In any particular case the transformation to the perpendicular form can be effected as follows;

Let the equation of the line, so expressed that the co-efficient of y is positive, be $ax + by + c = 0$. ....(1) (If it be not positive, it can be rendered so on multiplying both sides by -1).

Divide the equation throughout by $\sqrt{(a^2+b^2)}$ obtaining

$$\frac{a}{\sqrt{(a^2+b^2)}} x + \frac{b}{\sqrt{(a^2+b^2)}} y + \frac{c}{\sqrt{(a^2+b^2)}} = 0, \quad \ldots\ldots\ldots(2)$$

where the positive sign is taken with the radical in order that the coefficient of y namely $b/\sqrt{(a^2+b^2)}$ may be positive.

The normal form of equation (1) is thus

$$\frac{a}{\sqrt{(a^2+b^2)}}x+\frac{b}{\sqrt{(a^2+b^2)}}y=\frac{-c}{\sqrt{(a^2+b^2)}}\cdots\cdots\cdots(3)$$

The value of α is obtained from the relation $\tan\alpha=b/a$.

Cor. The length of the ⊥ from the origin on the line $ax+by+c=0$,(b positive), is therefore equal to $-c/\sqrt{(a^2+b^2)}$ the psitive value of the square root being taken.

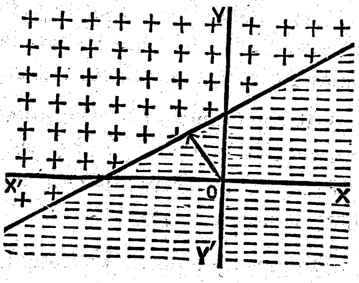
69. Sign of the Perpendicular.

It is important to pay attention to the sign of the perpendicular.

Let the equation to a st. line, with the coefficient of y rendered positive, be $ax+by+c=0$. If in this equation the constant b be negative, the origin lies on the negative side of the line. The perpendicular drawn from the origin on the line is therefore, in the direction "from negative to positive" that is, in the *increasing* direction, and hence, the perpendicular may, in accordence with the convention of Art 2, be taken as positive since it is measured in the increasing direction; and *vice versa*, if c is positive.

we may thus enunciate a convention for the singn of the perpendicular on a given st. line form any point.:

The perpendicular drawn to a line form a point on the negative side of the line is positive, while that drawn form a pint on the positive side is negative,



(Fig. 46)

With this convention we may say that the perpendicular from the orign on the line $ax+by+c=0$, (b positive), is $-c/\sqrt{(a^2+b^2)}$, both in sign and in magnitude.

The student should note that the perpendicular is a directed line-segment.

Example 1. *Express in the perpendicular form the equation to the line* $5x+12y+10=0$.

Divide the equation throughout by $\sqrt{(5^2+12^2)}$, or 13 ; we have $\frac{5}{13}x+\frac{12}{13}y+\frac{10}{13}=0$,
which is in the perpendicular form.

Example 2. *Throw into the normal form, the equation to the line* $20x-21y+58=0$ *and by considering the length of the perpendicular determine whether the origin lies on the positive or negative side of the line.*

Multiply the given equation by -1 to make the coefficient of y positive and then divide by $\sqrt{(20^2+21^2)}$ or 29 ; we thus obtain $-\frac{20}{29}x+\frac{21}{29}y-\frac{58}{29}=0$,
which is the normal form of the equation to the given line. The length of the ⊥ from the origin on the line is $+2$, showing thereby that the origin is on the *negative* side of the line.

Example 3. *Find the length of the perpendicular from the origin on the line* $x+3y+10=0$, *and show that the foot of the perpendicular lies on the circle* $x^2+y^2=10$.

Throwing the equation to the line into the normal form we have

$$\frac{x}{\sqrt{10}}+\frac{3y}{\sqrt{10}}+\sqrt{10}=0.$$

The length of the perpendicular from the origin is $-\sqrt{10}$. The distance of the foot of the ⊥ from the origin being numerically equal to $\sqrt{10}$, the foot lies on the circle of radius $\sqrt{10}$ and centre at the origin, i. e. lies on the circle

$$x^2+y^2=10.$$

Example 4. *Show that the two lines* $3x-4y+15=0$ *and* $24y-7x=75$ *are equidistant from the origin, and hence show that the two lines touch the circle* $x^2+y^2=9$.

Throwing the given equations into the perpendicular form with the co-efficient of y positive, we obtain

$$-\tfrac{3}{5}x+\tfrac{}{5}y=3, \quad \tfrac{7}{25}x+\tfrac{24}{25}y=3,$$

showing thereby that the lengths of the perpendiculars from the origin are 3 in both cases. The lines are thus equidistant from the origin.

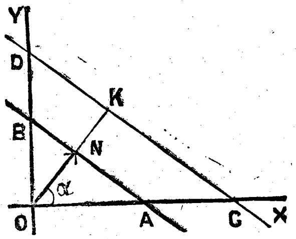
Further, the radius of the circle $x^2+x^2=9$ is 3 and the centre of the circle is at the origin. That is, the perpendicular from the centre of the circle to each of the two given lines is equal to the radius 3 ; hence the two lines are tangents to the circle, that is, they touch the circle.

Example 5. *Find the condition that the line* $y=mx+c$ *may be a tangent to the circle* $x^2+y^2=a^2$.

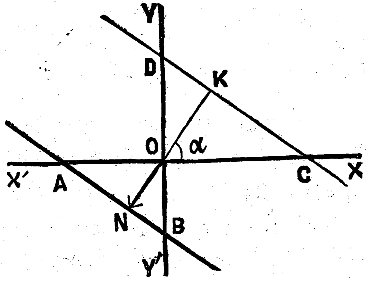
The centre of the circle is the origin (0, 0) whose perpendicular distance from the line $y=mx+c$ is $\pm c/\sqrt{(1+m^2)}$.

If the st. line is a tangent to the circle this perpendicular distance should be equal to the radius of the circle; that is, $a=\pm c/\sqrt{(1+m^2)}$ or $c=\pm a\sqrt{(1+m^2)}$.

70. Equation to Parallel Lines. Let AB, CD be two parallel lines (Fig. 47, 48) and ON, OK, the ⊥s from the origin on these lines. It is obvious that these two perpendiculars lie along the same line. Let the angle which this



(Fig. 47)



(Fig. 48)

line makes with the x-axis be α ; and let $ON=p$, $OK=p'$.

The equation to AB is $x\cos\alpha+y\sin\alpha=p$,

and that to CD $x\cos\alpha+y\sin\alpha=p'$.

Hence the equation to a line parallel to $x\cos\alpha+y\sin\alpha=p$ is of the form $x\cos\alpha+y\sin\alpha=p'$, where p' represents the length of the ⊥ from the origin on the parallel line.

Note. The position of the origin *w. r. t.* the lines AB, CD will be determined by the signs of p and p'. If, as in Fig. 47, p, p' are both positive, the origin lies on the negative side of both the lines while if they are both negative, the origin will lie on the positive side of both. If p, p' are of opposite signs, the origin will lie between the lines. The student is advised to draw a separate figure for each different case. In figure 48, p is negative and p' positive.

71. Distance between Parallel Lines.

If $x\cos\alpha+y\sin\alpha=p$, $x\cos\alpha+y\sin\alpha=p'$ be the equations of two parallel lines, it follows from the preceding Article, that the distance, NK, between the two parallel lines (Figs. 47, 48) is $(p'-p)$, both in sign and in magnitude.

The student should draw separate figures to illustrate the several cases that may arise depending on the signs of p and p', and verify that for all those cases the perpendicular distance measured from the first line to the second is $(p'-p)$.

If the equations to the two lines be given in the general form, the y should first be thrown into the normal form and the above result then applied.

Example 1. *Find the equation to a line through* (x_1, y_1) *parallel to the line* $x\cos\alpha+y\sin\alpha=p$.

Any line parallel to the given line is

$$x\cos\alpha+y\sin\alpha=p',$$

and if this should pass through (x_1, y_1) then p' is given by

$$x_1\cos\alpha+y_1\sin\alpha=p',$$

so that the equation to the parallel assumes the form

$$x\cos\alpha+y\sin\alpha=x_1\cos\alpha+y_1\sin\alpha.$$

An alternative method is to apply the result of Art. 55, whereby we immediately get the equation to the line passing through (x_1, y_1) and ∥ to the given line as

$$(x - x_1)\cos\alpha + (y - y_1)\sin\alpha = 0.$$

Example 2. *Find the distance between the two parallel lines* $5x - 12y + 19 = 0,\quad 12y - 5x + 33 = 0.$

Reduce the given equations to the normal form, keeping the co-efficient of y positive in both the cases; we thus have

$$-\tfrac{5}{13}x + \tfrac{12}{13}y - \tfrac{19}{13} = 0, \qquad -\tfrac{5}{13}x + \tfrac{12}{13}y + \tfrac{33}{13} = 0.$$

∴ The perpendicular from the origin on the first line is $\frac{19}{13}$; and that on the second line is $-\frac{33}{13}$.

The distance between the two lines is, therefore,

$$\tfrac{19}{13} + \tfrac{33}{13} = 4.$$

Note. The length of the perpendicular can be obtained directly as $\frac{19}{13}$ and $-\frac{33}{13}$ by applying the result of Art. 69, and the subsequent steps are the same as before.

Exercise XVI

Find the equation to the st. lines having:

1. ⊥ from origin $=4$, inclination of ⊥ to the x-axis 45°.

2. ⊥ from origin $= -3$, inclination of ⊥ to the x-axis 30°.

3. ⊥ from origin $= -\sqrt{3}$, inclination of ⊥ to x-axis 120°.

4. ⊥ from origin $= 2\frac{1}{5}$, slope of the ⊥ equals $\frac{3}{4}$.

Throw the equations of the following lines into the normal form and find the sign and magnitude of the ⊥ from the origin in each case:

5. $3x + 4y - 10 = 0.$ **6.** $12x - 5y + 39 = 0.$

7. $15x + 8y = 51.$ **8.** $9x - 40y = 16\frac{2}{5}.$

9. Find the equations to the lines ∥ to $x\cos\alpha + y\sin\alpha = p$ and passing through

(1) $(4, 7)$. (2) $(-2, -3)$. (3) $(-p\cos\alpha, -p\sin\alpha)$.

10. Find the distance between the following pairs of parallel lines;

(1) $3x - 4y + 12 = 0,\quad 4y - 3x + 8 = 0.$
(2) $5x + 12y + 7 = 0,\quad 5x + 12y = 19.$
(3) $15x - 8y + 30 = 0,\quad 8y - 15x + 38 = 0.$
(4) $9x + 40y + 27 = 0,\quad 9x + 40y - 14 = 0.$
(5) $2x - 3y + 4 = 0,\quad 4x - 6y - 5 = 0.$

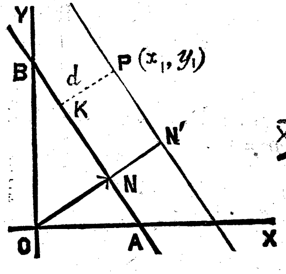
11. Show that the lines $4x - 3y + 15 = 0$, $4x + 3y - 15 = 0$ are tangents to the circle of radius3 and centre at the origin.

12. Show that the two straight lines $3x - 4y + 3\frac{1}{3} = 0$ and $5x + 12y + 8\frac{2}{3} = 0$ touch the circle $9(x^2 + y^2) = 4$.

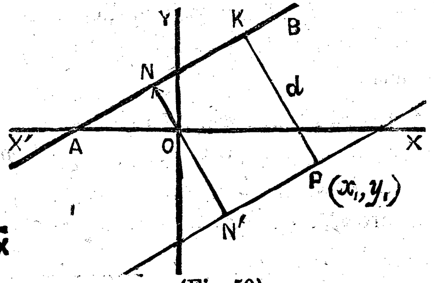
13. Show that the origin is the in-centre of the triangle formed by the st. lines $3x + 4y = 15$, $7x - 24y + 75 = 0$, $8x + 15y + 51 = 0$. Also find the equation to the in-circle.

14. Show that the perpendicular distance of the origin from the line joining the two points (x_1, y_1), (x_2, y_2) is $(x_1y_2 - x_2y_1)/\sqrt{[(x_1 - x_2)^2 + (y_1 - y_2)^2]}$. (*Vide* Ex. XII, Q. 17.)

72. Length of Perpendicular from a Point on a Line. Let P (x_1, y_1) be a point (Figs. 49, 50) the length of the ⊥ from which to the line $x\cos\alpha + y\sin\alpha - p = 0$ ..(1) is required.



(Fig 49)



(Fig 50)

The equation to a st. line through (x_1, y_1) ∥ to the given line is $x\cos\alpha + y\sin\alpha - (x_1\cos\alpha + y_1\sin\alpha) = 0. \ldots\ldots (1)$
∴ The length of the perpendicular from the origin on this line is $\quad x_1\cos\alpha + y_1\sin\alpha,$
due regard being paid to the sense of the perpendicular. But the length of the ⊥ from the origin on line (1) is p.
∴ The distance of line (1) from line (2) is

$$p - x_1\cos\alpha - y_1\sin\alpha$$

or $-(x_1\cos\alpha + y_1\sin\alpha - p)$, and this is the length of the ⊥ from (x_1, y_1) on the given line,

Thus, *if the equation of a line is expressed in the form* $x\cos\alpha + y\sin\alpha - p = 0$ *the length of the perpendicular to the line from any point* (x_1, y_1) *is obtained by substituting the co-ordinate of the point in the expression on the left hand side of the equation and reversing the sign.*

Applying this result, the ⊥ distance of the origin from the line $x\cos\alpha + y\sin\alpha - p = 0$ is p. Note that the expression gives both in sign and in magnitude the perpendicular distance *from the point to the line.* If the distance from the line to the point is required, the sign has to be altered.

73. Length of Perpendicular : Alternative Methods.

The length of the perpendicular on a line from any point may also be derived in any one of the following alternative methods.

1. Shift the origin to the point (x_1, y_1); the equation to the line $\quad x\cos\alpha + y\sin\alpha - p = 0$
changes to $\quad (X + x_1)\cos\alpha + (Y + y_1)\sin\alpha - p = 0,$
where (X, Y) are the co-ordinates of a point *w. r. t.* the new origin (x_1, y_1);

$$\therefore X\cos\alpha + Y\sin\alpha + (x_1\cos\alpha + y_1\sin\alpha - p) = 0,$$

which shows that the length of the ⊥ from the new orig (x_1, y_1) on the given line is $-(x_1\cos\alpha + y_1\sin\alpha - p)$.

2. The distance of $P(x_1, y_1)$ from the line AB, (Fig. 49) is $\quad d = N'N = ON - ON'.$

But ON' = the projection of OP on ON
= the sum of the projections of OM, MP on ON,
(where PM is the ordinate of P)

$$= x_1\cos\alpha + y_1\sin\alpha,$$

Hence $d = ON - ON' = p - (x_1\cos\alpha + y_1\sin\alpha)$

$$= -(x_1\cos\alpha + y_1\sin\alpha - p).$$

3. Let $OP = r$, $\angle XOP = \theta$.

Then $\quad ON' = r\cos(\theta - \alpha)$

$$= r\cos\theta\cos\alpha + r\sin\theta\sin\alpha$$
$$= x_1\cos\alpha + y_1\sin\alpha,$$

whence, as before, $d = ON - ON' = p - (x_1\cos\alpha + y_1\sin\alpha)$

$$= -(x_1\cos\alpha + y_1\sin\alpha - p).$$

4. If MP the ordinate of P, meets the line AB at Q, then

$$MQ = MP + PQ,$$
$$\therefore PQ = MQ - MP.$$

But $MP = y_1$, $\quad OM = x_1$, $\quad NR$ = the ordinate of Q
$= (p - x_1\cos\alpha)/\sin\alpha$, (∵ Q lies on $x\cos\alpha + y\sin\alpha - p = 0$),

$$\therefore PQ = (p - x_1\cos\alpha)/\sin\alpha - y_1$$
$$= (p - x_1\cos\alpha - y_1\sin\alpha)/\sin\alpha.$$

Also, the angle between MP, PK is equal to the angle between the parallel lines OM, ON', which is seen to be the compliment of α.

$$\therefore PK = PQ\cos KPM$$
$$= \frac{p - x_1\cos\alpha - y_1\sin\alpha}{\sin\alpha}\cdot\cos(90^\circ - \alpha)$$
$$= p - x_1\cos\alpha - y_1\sin\alpha.$$

74. Length of Perpendicular : General Form.

If the equation to the st. line be given in the general form $ax + by + c = 0$, it must first be expressed in the corresponding perpendicular form, namely,

$$\frac{a}{\sqrt{a^2+b^2}}x+\frac{b}{\sqrt{a^2+b^2}}y+\frac{c}{\sqrt{a^2+b^2}}=0$$

Hence, the length of the perpendicular from the point (x_1, y_1) to the given line is $-(ax_1+by_1+c)/\sqrt{a^2+b^2}$.

As an immediate corollary of this we get the length of the ⊥ from the origin on the line $ax+by+c=0$ to be $-c/\sqrt{(a^2+b^2)}$, which agrees with the result of Art. 68, Cor.

75. Length of Perpendicular: General Form. Alternative Methods.

1. The length of the ⊥ to the line $ax+by+c=0$ from the point P (x_1, y_1) may be obtained by shifting the origin to P, when the equation to the line becomes

$$a(X+x_1)+b(Y+y_1)+c=0,$$

or
$$aX+bY+(ax_1+by_1+c)=0.$$

∴ The length of the perpendicular from the new origin P to the line is $-(ax_1+by_1+c)/\sqrt{a^2+b^2}$, (Art. 68, Cor.).

2. Another independent method is to follow the procedure adopted in Art. 73, 4.

3. The equation to the line through (x_1, y_1) parallel to

the line $\qquad ax+by+c=0$

is (Art. 55) $\qquad ax+by-(ax_1+by_1)=0.$

The lengths, p, p', of the perpendiculars from the origin on the above lines are given by

$$p=-\frac{c}{\sqrt{a^2+b^2}},\ p'=-\frac{-(ax_1+by_1)}{\sqrt{a^2+b^2}}=\frac{ax_1+by_1}{\sqrt{a^2+b^2}}.$$

But the length of the ⊥ from (x_1, y_1) on $ax+by+c=0$ is $p-p'$, and is, therefore, equal to $\quad -\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$.

4. The following independent method of deducing the length of the perpendicular is of interest.

Let (x_2, y_2) be the foot of the ⊥ on the line $ax+by+c=0$ from the point (x_1, y_1). (The student is advised to draw a diagram to illustrate this.)

The length p, of the ⊥ is the distance of (x_2, y_2) from (x_1, y_1);

$$\therefore\ p=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\ \ldots\ldots\ldots\ldots(1)$$

The equation to the ⊥ on $ax+by+c=0$ from (x_1, y_1) is, by Art. 56, $\qquad b(x-x_1)-a(y-y_1)=0.\ \ldots\ldots\ldots\ldots(2)$

The point (x_2, y_2) lies on this perpendicular,

$$\therefore\ b(x_2-x_1)-a(y_2-y_1)=0.\ \ldots\ldots\ldots\ldots(3)$$

Since (x_2, y_2) also lies on the given line, we have

$$ax_2+by_2+c=0;$$

subtracting ax_1+by_1+c from both sides of this equation,

$$a(x_2-x_1)+b(y_2-y_1)=-(ax_1+by_1+c).\ \ldots\ldots\ldots(4)$$

Square both sides of equations (3) and (4) and add:

$$(a^2+b^2)\{(x_2-x_1)^2+(y_2-y_1)^2\}=(ax_1+by_1+c)^2;$$

therefore, by (1), $(a^2+b^2)p^2=(ax_1+by_1+c)^2$,

whence $\qquad p=\pm\dfrac{ax_1+by_1+c}{\sqrt{(a^2+b^2)}}$.

5. If $P(x_1, y_1)$ is any point and $ax+by+c=0$ any straight line then the area of the triangle ABP is $\frac{1}{2}$ AB. p. (Fig. 49) A is the point $(-c/a, 0)$, B is the point $(0, -c/b)$.

$$\therefore\ AB^2=\frac{c^2}{a^2}+\frac{c^2}{b^2}=\frac{(a^2+b^2)c^2}{a^2b^2}.$$

$$\therefore\ \triangle=\tfrac{1}{2}\frac{c\sqrt{(a^2+b^2)}}{ab}\times p.$$

But the area of the triangle formed by the points A, B, P is

$$\tfrac{1}{2}(cx_1/b+cy_1/a+c^2/ab)\quad\text{or}\quad\tfrac{1}{2}c(ax_1+by_1+c)/ab.$$

Equating the two expressions for △, we obtain

$$p\sqrt{(a^2+b^2)}=ax_1+bx_1+c,$$

whence $\qquad p=(ax_1+by_1+c)/\sqrt{(a^2+b^2)}$.

6. Let (x_2, y_2), (x_3, y_3) be any two points on the line $ax+by+c=0$, and d the distance between them. The area of the triangle formed by these two points and the given point (x_1, y_1) is

$$\tfrac{1}{2}[x_1(y_2-y_3)-y_1(x_2-x_3)+(x_2y_3-x_3y_2)],$$

and is also given by the relation

$$\triangle=\tfrac{1}{2}pd \quad \text{or} \quad 2\triangle=pd.$$

But since (x_2, y_2), (x_3, y_3) lie on $ax+by+c=0$, we have

$$ax_2+by_2+c=0, \quad ax_3+by_3+c=0, \quad \text{whence}$$

$$\frac{a}{y_2-y_3}=\frac{b}{x_3-x_2}=\frac{c}{x_2y_3-x_3y_2}=\frac{1}{k}, \text{ say.}$$

$$\therefore\ y_2-y_3=ak, \quad x_3-x_2=bk, \quad x_2y_3-x_3y_2=ck.$$

Substituting these values, the area becomes equal to

$$\tfrac{1}{2}[akx_1+bky_1+ck].$$

Also $\quad d^2=a^2k^2+b^2k^2=k^2(a^2+b^2)$.

$$\text{Hence} \quad p=\frac{2\triangle}{d}=\frac{k(ax_1+by_1+c)}{k\sqrt{(a^2+b^2)}}=\frac{ax_1+by_1+c}{\sqrt{(a^2+b^2)}}.$$

Note. This formula gives the *numerical* value of the length of the perpendicular.

The distance between any two parallel lines $ax+by+c=0$ and $ax+by+c'=0$ can be derived by employing the above formula for the length of the perpendicular thus:

If (x_1, y_1) be any point in the first line, then its distance from the second line is

$$(ax_1+by_1+c')/\sqrt{(a^2+b^2)}=(-c+c')/\sqrt{(a^2+b^2)}.$$

This is the distance between the two ‖ lines.

Example 1. *Find the length of the perpendicular from the point* $(-3, 2)$ *to the line* $5x-12y-39=0$.

Making the co-efficient of y positive and expressing the given equation in the perpendicular form, we get

$$-\tfrac{5}{13}x+\tfrac{12}{13}y+3=0.$$

The length of the perpendicular from $(-3,2)$ on this line is

$$-[-\tfrac{5}{13}.(-3)+\tfrac{12}{13}.2+3]=-6.$$

Example 2. *Find the distance of the point* $(4, 3)$ *from the line* $8y-15x+2=0$, *and find whether this point lies on the same side of the line as the origin or on the opposite side.*

The ⊥ distance of the origin from the given line is $-\frac{2}{17}$, the distance of $(4, 3)$ from the given line is

$$-\frac{8.3-15.4-2}{17}=+2$$

Hence the point $(4, 3)$ lies on that side of the given line opposite the origin; or, as we may put it otherwise, the point $(4, 3)$ lies on the non-origin side of the line.

Example 3. *By considering the lengths of the perpendiculars from* (x_1, y_1), (x_2, y_2) *on the line* $ax+by+c=0$, *determine whether the two points lie on the same side or on opposite sides of the given line.*

The lengths of the perpendiculars from (x_1, y_1), (x_2, y_2) on the given line are

$$-(ax_1+by_1+c)/\sqrt{a^2+b^2} \quad \text{and} \quad -(ax_2+by_2+c)/\sqrt{a^2+b^2}.$$

$\therefore$ The ratio of the lengths $=(ay_1+by_1+c)/(ax_2+by_2+c)$.

If the ratio of the lengths is positive the points lie on the same side of the line, while if the ratio is negative the points lie on opposite sides of the line. Hence, the two points lie on the same side or on opposite sides of the line according as ax_1+by_1+c and ax_2+by_2+c are of like or unlike signs.

Example 4. *Show that the equation to the locus of a point which is at a distance* 2 *units from the line* $24x+7y=35$ *consists of two lines whose equations are* $24x+7y+15=0$ *and* $24x+7y=85$.

Let (x_1, y_1) be a point satisfying the given condition; the length of the perpendicular from the point (x_1, y_1) on the line $24x+7y-35=0$ is $-(\frac{24}{25}x_1+\frac{7}{25}y_1-\frac{35}{25})$.

But the distance is 2, in magnitude ,

$$\therefore\ -\left(\tfrac{24}{25}x_1+\tfrac{7}{25}y_1-\tfrac{35}{25}\right)=\pm 2,$$

or $$24x_1+7y_1-35\pm 50=0.$$

∴ Equation to the locus of (x_1, y_1) consists of the two lines $$24x+7y+15=0 \quad\text{and}\quad 24x+7y=85.$$

Exercise XVII

Find the length of the perpendicular from the given point to the given line in each of the following cases:

1. from (3, 4) to $3x+4y-5=0$.
2. from (1, −2) to $5x-12y+10=0$.
3. from$(-\frac{1}{2}, \frac{1}{3})$ to $8x+15y-18=0$.
4. from$(1, 4\frac{1}{2})$ to $24x-7y-5=0$.
5. from (−1, −2) to $x-3y+5=0$.
6. from $(3, -\frac{9}{4})$to $4x+4y-19=0$.

In each of the following cases find the length of the perpendicular from the given point to the given line and hence determine whether the given point lies on the origin side or the non-origin side of the line :

7. from (1, 2) to $4x-3y+1=0$.
8. from (−5, 0) to $8x+15y+6=0$.
9. from (1, 1) to $2x-y-3=0$
10. from $(-1, \frac{1}{2})$ to $9x+40y+71=0$.
11. Find the lengths of the perpendiculars from (2, 3) and (4, 7) to the line $2x-y-2=0$ and determine whether the points are on the same side or on opposite sides of the line. What is the inference from the results?
12. By calculating the lengths of the perpendiculars from (1, 3) and (3, 1) to the line $4x+5y=20$, find whether the points lie on the same side or on opposite sides of the line.
13. Show that the point (2, −3) lies outside the parallel lines $2x+y=3$, $2x+y-4=0$.

14. Show that the origin and the point (6, 1) lie within vertically opposite angles formed by the lines $y-x+4=0$, $2x+y-4=0$.
15. Find the equation to the locus of a point the sum of the squares of whose distances from the lines $2x+y+1=0$ $x+2y-1=0$ is 3.
16. Find the equation to the locus of a point which is such that its distance from (3, 4) is equal to its distance from the line $x-y+5=0$.
17. P is a variable point whose distance from the line $x+2y=0$ is $\sqrt{2}$ times its distance from (1, 1); show that the equation to the locus of P is $$9x^2+6y^2-4xy-20x-20y+20=0.$$
18. Show that the equation to the locus of a point whose distance from the point (2, 1) is twice its distance from the line $x-2y+3=0$ is $x^2+16xy-11y^2-44x+38y-11=0$.

76. Bisectors of the Angles between two Lines

Let $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ be two intersecting st. lines, (the condition that they should intersect is that $a_1:a_2 \neq b_1:b_2$), the equations being so expressed as to make the co-efficient of y positive in both the cases and let (h, k) be any point equidistant from the lines.

The length of the perpendicular from (h, k) to the first line is $-(a_1h+b_1k+c_1)/\sqrt{a_1^2+b_1^2}$ and that to the second line is $-(a_2h+b_2k+c_2)/\sqrt{a_2^2+b_2^2}$.

Since the point (h, k) is equidistant from the two lines, the lengths of the two ⊥s are equal in magnitude; that is

$$(a_1h+b_1k+c)/\sqrt{a_1^2+b_1^2}=(a_2h+b_2k+c_2)/\sqrt{a_2^2+b_2^2},$$

or $$(a_1h+b_1k+c_1)/\sqrt{a_1^2+b_1^2}=-(a_2h+b_2k+c_2)/\sqrt{a_2^2+b_2^2}.$$

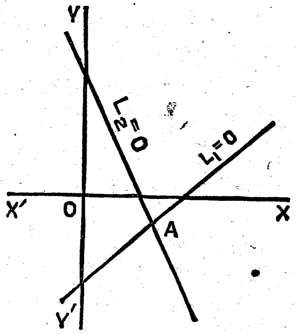
∴ The point (h, k) lies on one or the other of the two st lines $$\frac{a_1x+b_1y+c_1}{\sqrt{(a_1^2+b_1^2)}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{(a_2^2+b_2^2)}},$$

The student, doubtless, has known from Geometry that the locus of a point equidistant from two intersecting lines is the pair of bisectors of the angles between the two given lines. The above equations, therefore, represent the bisectors of the angles between the lines

$$a_1x+b_1y+c_1=0 \quad \text{and} \quad a_2x+b_2y+c_2=0.$$

77. Discrimination of the Bisectors

Let $L_1 \equiv a_1x+b_1y+c_1=0$, $L_2 \equiv a_2x+b_2y+c_2=0$ be the equations to the two given st. lines where b_1, b_2, the coefficients of y, are positive. Fig 51 represents the case where c_1 is positive and c_2 negative; the student is advised to draw figures for the other cases.



(Fig. 51)

Of the four angles formed by the intersection of the lines $L_1=0$, $L_2=0$, let us consider that angle in which the origin lies. The origin, in Fig. 51, lies on the positive side of $L_1=0$, and on the negative side of $L_2=0$, and hence the perpendiculars from the origin to the two lines are of opposite signs. And this is true of any other point within the angle considered. Thus, if (x_1, y_1) be any point on the bisector of that angle in which the origin lies, then the perpendiculars from (x_1,y_1) on the two lines should be equal in magnitude but opposite in sign; i.e. $(a_1x_1+b_1y_1+c_1)/\sqrt{a_1^2+b_1^2}=-(a_2x_1+b_2y_1+c_2)/\sqrt{a_2^2+b_2^2}$

Hence the equation to the locus of (x_1,y_1), in other words, the equation to the bisector of that angle which *contains the origin* is

$$\frac{a_1x+b_1y+c_1}{\sqrt{(a_1^2+b_1^2)}}=-\frac{a_2x+b_2y+c_2}{\sqrt{(a_2^2+b_2^2)}}.$$

and that to the other bisector is

$$\frac{a_1x+b_1y+c_1}{\sqrt{(a_1^2+b_1^2)}}=\frac{a_2x+b_2y+c_2}{\sqrt{(a_2^2+b_2^2)}}.$$

If c_1, c_2 be of the same sign, then the right sides of the above two equations change sign.

Example 1. *Show that the point* (2, 3) *lies on one of the bisectors of the angles between the st. lines* $3x+4y+2=0$ *and* $5y=12x+43$.

The lengths of ⊥s from (2, 3) on the two lines are

$$-\frac{3.2+4.3+2}{5} \text{ and } -\frac{5.3-12.2-43}{12}, \text{ i.e. } -4 \text{ and } 4.$$

Hence (2, 3) is equidistant from the two lines and therefore lies on a bisector of the angle between them.

Example 2. *Find the equation to the bisector of the angles between the lines* $4x=3y+7$ *and* $3x+4y=2$, *and find the bisector of that angle which contains the origin.*

The equations to the two bisectors are

$$\frac{-4x+3y+7}{5}=\pm\frac{3x+4y-2}{5},$$

the lower sign yielding the equation to the bisector of that angle containing the origin, since the constants are of opposite signs. Simplifying the above equations, we get

$$7x+y-9=0, \qquad -x+7y+5=0,$$

of which the latter equation represents the bisector of that angle which contains the origin.

Example 3. *Find the in-centre of the triangle whose sides are the st. lines* $3x+4y=3$, $5x-12y=5$, $y=5$.

By drawing a figure to represent the above lines, it is easily verified that if $I(h, k)$ be the in-centre, the lengths of the perpendiculars from I on the first two lines are negative, while that on the third line is negative. Equating the lengths of the perpendiculars, we have

$$\frac{3h+4k-3}{5}=\frac{12k-5h+5}{13}=-(k-5),$$

or $\quad 3h+4k-3=25-5k, \quad 12k-5h+5=65-13k,$

i. e. $\quad 3h+9k=28, \quad -5h+25k=60.$

Solving for h and k, we obtain $h=1\frac{1}{3}$, $k=2\frac{2}{3}$,

Hence the in-centre is the point $(1\frac{1}{3}, 2\frac{2}{3})$.

78. Symmetrical Form of the Equation to a Straight Line. Let PQ be any st. line passing through $P(x_1, y_1)$ and inclined at θ to the x-axis. Let $PQ=r$, where $Q\equiv(x, y)$; then considering the projection of PQ on OX

$$r\cos\theta=MN=x-x_1;$$

similarly, $\quad r\sin\theta=RQ=y-y_1.$

Hence
$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r,$$

which is the equation satisfied by the co-ordinates of any point on the line PQ.

This form of the equation to the st.line is called the **symmetrical form** or the **parametric form.**

One feature of this form is that it makes use of the distance of the point (x, y) from the fixed point (x_1, y_1).

Corollary. The co-ordinates of a variable point on the line through (x_1, y_1) and inclined to the x-axis at an angle θ may be expressed as $x=x_1+r\cos\theta$, $y=y_1+r\sin\theta$, where r is the distance of the point from (x_1, y_1).

Example. PB *is a st. line of slope* $\frac{3}{4}$ *passing through* A $(-3, 2)$; *find the co-ordinates of* B *given that* $AB=15$.

Let the co-ordinates of B be (x, y) and let the inclination of the line to the x-axis be θ.

Since $\tan\theta=\frac{3}{4}$, we get $\sin\theta=\frac{3}{5}$, $\cos\theta=\frac{4}{5}$. Also, since the magnitude of AB is 15, $x=x_1\pm15\cos\theta$, $y=y_1\pm15\sin\theta$,

$$\therefore\ x=-3\pm15.\tfrac{4}{5}=9 \text{ or } -15,\ y=2\pm15.\tfrac{3}{5}=11 \text{ or } -7.$$

$\therefore$ The co-ordinates of B are $(9, 11)$ or $(-15, -7)$.

Exercise XVIII

Find the equations to the bisectors of the angles between the following pairs of lines, placing first the bisector of that angle in which the origin lies:

1. $3x-4y+1=0, \quad 3x+4y+9=0.$
2. $3x-2y+1=0, \quad 2x-3y+4=0.$
3. $24x+7y-31=0, \quad 4x+3y-7=0.$
4. $7x+y+3=0; \quad y=y+5.$
5. $3x-4y+6=0, \quad 12x+5y+10=0.$
6. $4x-2y+3=0, \quad 2x-4y+7=0.$
7. Show that the lines $4x+3y+3=0$, $12x+5y+13=0$ are tangents to the circle with centre at $(2, 3)$ and radius 4.
8. Find the co-ordinates of the centre of the inscribed circle of the triangle the equations to whose sides are $3x-4y=0$, $7x-24y=0$, $5x-12y-36=0$.
9. Find the in-centre of the triangle formed by the lines $3x-4y+10=0$, $5x-12y-26=0$, $4x+3y+10=0$.
10. Find the in-centre of the triangle formed by the lines $2x+y=8$, $2x+11y+32=0$, $x+1=2y$.
11. A is the point $(3, -1)$ and B lies on the line $4x-3y-1=0$, if B has gradient $\frac{5}{12}$, find the length AB.
12. A st. line is drawn through $P(h, k)$ inclined at θ to the x-axis and meeting the line $ax+by+c=0$ at a distance r from P. Show that $r=-(ah+bk+c)/(a\cos\theta+b\sin\theta)$.

Hence deduce the length of the perpendicular from (h, k) on the line $ax+by+c=0$.

CHAPTER IX

Concurrence of Lines. Systems of Lines

79. Line through Intersection of two Lines

Let
$$L \equiv ax+by+c=0, \quad \ldots\ldots\ldots\ldots (1)$$
$$L' \equiv a'x+b'y+c'=0, \quad \ldots\ldots\ldots\ldots (2)$$
be the equations to two st. lines which intersect at the point P (x_1, y_1).

Since (x_1, y_1) lies on each of the lines, we have
$$ax_1+by_1+c=0, \quad \ldots\ldots\ldots\ldots (3)$$
$$a'x_1+b'y_1+c'=0 \quad \ldots\ldots\ldots\ldots (4)$$
From these equations, we find, as in Art. 61, that
$$x_1=(bc'-b'c)/(ab'-a'b), \quad y_1=(ca'-c'a)/(ab'-a'b).$$
Now consider the equation $L+\lambda L'=0$

i.e. $$(ax+by+c)+\lambda(a'x+b'y+c')=0, \quad \ldots\ldots (5)$$
λ being any arbitrary constant. This is an equation of the first degree in x and y, and therefore represents a st. line. Further, equation (5) is satisfied by (x_1, y_1), the co-ordinates of the point of intersection of the lines (1) and (2), for, on substitution of (x_1, y_1) in (5), we get
$$(ax_1+by_1+c)+\lambda(a'x_1+b'y_1+c')=0,$$
an equality that is true in virtue of the relations (3) and (4).

Therefore the st. line represented by (5) passes through (x_1, y_1).

Hence we conclude that

if $L \equiv ax+by+c=0$, $L'=a'x+b'y+c'=0$ *be the equations to two straight lines, then* λ *being any arbitrary constant, the equation* $L+\lambda L'=0$, *represents any straight line passing through the point of intersection of the lines* $L=0$, $L'=0$.

Note 1. For different values of λ, equation (5) represents different st. lines. All these lines are said to constitute a system of lines.

2. Given one other condition satisfied by any line of the system represented by (5), we can find the corresponding value of λ and thus determine the line completely.

Example 1. *Find the equation to the straight line joining the origin and the point of intersection of the lines* $5x+y-3=0$, $x-y+1=0$.

The equation of the required line is of the form
$$(5x+y-3)+\lambda(x-y+1)=0,$$
where λ is any arbitrary constant. If this line passes through the origin, the constant term in the above equation should be zero; i.e. $\lambda-3=0$ or $\lambda=3$.

$\therefore$ The required equation to the line is
$$(5x+y-3)+3(x-y+1)=0, \quad \text{or} \quad 4x-y=0.$$

Example 2. *Find the equation to the st. line passing through the intersection of* $x+y=1$ *and* $2x=y-4$ *and having slope* 4.

The equation to any st. line passing through the point of intersection of the two given lines is
$$(x+y-1)+k(2x-y+4)=0.$$
The slope of this line is $(2k+1)/(k-1)$, and must therefore be equal to 4.
$$\therefore\ 2k+1=4(k-1), \quad \text{or} \quad k=\tfrac{5}{2}.$$
$\therefore$ The required equation is $(x+y-1)+\frac{5}{2}(2x-y+4)=0$,
which simplifies to $\quad 4x-y+6=0.$

Example 3. *Find the equation to the st. line passing through the intersection of the lines* $x+y+4=0$ *and* $4x-3y=12$ *and perpendicular to the line* $2y=7-3x$.

The equation to any line passing through the intersection of the first two lines is $(4x-3y-12)+k(x+y+4)=0$.
The slope of this line is $(4+k)/(3-k)$.

If this is to be ⊥ to the line $2y=7-3x$, the product of the slopes $=-1$,

i. e. $$\frac{4+k}{3-k}\times(-1\tfrac{1}{2})=-1,$$

or $$3(4+k)=2(3-k) \text{ which gives } k=-\tfrac{6}{5}.$$

Substituting this value of k, we have

$$5(4x-3y-12)-6(x+y+4)=0.$$

which reduces to $2x-3y-12=0$.

This, therefore, is the equation to the st. line passing through the intersection of the lines $x+y+4=0$ and $4x-3y=12$ and perpendicular to the line $2y=7-3x$.

Example 4. *Find the equations to the st. lines passing through the intersection of* $6x+5y+4=0$ *and* $2x+y+2=0$ *and which are at a distance* 4 *from the point* (2, 3).

The equation to any line passing through the intersection of the two given lines is

$$(6+5y+4)+\lambda(2x+y+2)=0. \quad \ldots\ldots\ldots\ldots(1)$$

or $$(6+2\lambda)x+(5+\lambda)y+(4+2\lambda)=0.$$

But the distance of this line from (2, 3) is 4,

$$\therefore \frac{(6+2\lambda).2+(5+\lambda).3+(4+2\lambda)}{\sqrt{(6+2\lambda)^2+(5+\lambda)^2}}=4.$$

or $$(9\lambda+31)^2=16\{(6+2\lambda)^2+(5+\lambda)^2\}$$

i.e. $$\lambda^2+14\lambda-15=0, \qquad \therefore \lambda=1 \text{ or } -15$$

Substitute these values of λ in (1) and simplify; we obtain

$$4x+3y+3=0 \quad \text{and} \quad 12x+5y+13=0$$

as the equations to the two st. lines passing through the intersection of the given lines and distant 4 from (2, 3).

80. Concurrence of Three Lines.

Let
$$a_1x+b_1y+c_1=0, \quad \ldots\ldots\ldots\ldots(1)$$
$$a_2x+b_2y+c_2=0, \quad \ldots\ldots\ldots\ldots(2)$$
$$a_3x+b_3y+c_3=0, \quad \ldots\ldots\ldots\ldots(3)$$

be three straight lines, no two of which are parallel.

We have already seen that the lines (1) and (2) meet at the point (x', y') where

$$x'=(b_1c_2-b_2c_1)/(a_1b_2-a_2b_1), \quad y'=(c_1a_2-c_2a_1)/(a_1b_2-a_2b_1).$$

If the st. line (3) also passes through (x', y'), we must have

$$a_3x'+b_3y'+c_3=0;$$

substituting herein the values of x', y' and simplifying,

$$a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)=0.$$

This, therefore, is the condition that the three given lines may be concurrent.

81. Condition for Concurrence : Another Form.

The condition for the concurrency of three lines may be expressed in another form.

If l, m, n are any constants other than zero, such that

$$l(a_1x+b_1y+c_1)+m(a_2x+b_2y+c_2)+n(a_3x+b_3y+c_3)\equiv 0 \quad ..(4)$$

for all values of x and y, then three straight lines represented by (1), (2) and (3) are concurrent.

If (h, k) be the co-ordinates of the point of intersection of the lines (2) and (3), we have

$$a_2h+b_2k+c_2=0, \qquad a_3h+b_3k+c_3=0.$$

But by (4)

$$l(a_1h+b_1k+c_1)+m(a_2h+b_2k+c_2)+n(a_3h+b_3k+c_3)=0.$$
$$\therefore l(a_1h+b_1k+c_1)=0,$$
$$\therefore a_1h+b_1k+c_1=0, \quad \text{since } l\neq 0,$$

that is, the line $a_1x+b_1y+c_1=0$ passes through (h, k).

Thus the three given st. lines pass through (h, k) and are therefore concurrent.

82. Condition for Concurrence : Alternative Method.

The condition for the concurrence of the lines may also be established in the following manner:

Any line passing through the intersection of (1) and (2) is given by $(a_1x+b_1y+c_1)+\lambda(a_2x+b_2y+c_2)=0, \ldots\ldots\ldots(5)$ where λ is any arbitrary constant.

For some value of λ, this equation will represent the same line as (3), provided that line (3) passes through the intersection of (1) and (2). That is, if the three lines are concurrent, equation (5) represents, for some particular value of λ, the same line as (3).

The condition for this, by Art. 46. Cor. 2 that

$$\frac{a_1+\lambda a_2}{a_3}=\frac{b_1+\lambda b_2}{b_3}=\frac{c_1+\lambda c_2}{c_3}.$$

The values of λ obtained from the two equations will be the same, i.e. $\dfrac{a_3b_1-a_1b_3}{a_2b_3-a_3b_2}=\dfrac{b_3c_1-b_1c_3}{b_2c_3-b_3c_2}$,

cross-multiplying and simplifying, we get

$$\Sigma\, a_1(b_2c_3-b_3c_2)=0,$$

a condition identical with that of Art. 80.

Example 1. *Show that the three lines* $2x-y+1=0$, $x+y-1=0$, $x+5=5y$ *are concurrent.*

If it is possible to find l, m, n such that

$$l(2x-y+1)+m(x+y-1)+n(x-5y+5)=0$$

for all values of x and y, then the three lines are concurrent.

l, m, n must therefore, be found to satisfy the relations

$$2l+m+n=0, \quad -l+m-5n=0, \quad l-m+5n=0,$$

$\therefore\ l=-2,\ m=3,\ n=1$, are one set of values satisfying the above conditions.

We verify that $-2(2x-y+1)+3(x+y-1)+(x-5y+5)=0$.

$\therefore$ The three given lines are concurrent.

Example 2. *Show that the three lines* $x+2y-5=0$, $2x-3y+4=0$, $4y=9x-1$ *meet in a point.*

Any line through the meet of the first two lines is

$$(x+2y-5)+k(2x-3y+4)=0,$$

k being any arbitrary constant.

If the three given lines are concurrent, this equation must, for some value of k, represent the same line as the equation $\quad 4y-9x+1=0$;

$$\therefore\ \frac{1+2k}{-9}=\frac{2-3k}{4}=\frac{4k-5}{1}.$$

These equations are consistent, since both give for k the same value, namely, $\frac{22}{19}$.

Hence, the three lines meet in a point.

Example 3. *Show that the line joining the points* $(2, -1)$ $(-3, 4)$ *is concurrent with the lines* $3x+y+1=0$, $2x-y+4=0$.

Any line through the intersection of the two given lines is $\quad (3x+y+1)+\lambda(2x-y+4)=0.$

If this passes through $(2, -1)$, we have

$$(3.2-1+1)+\lambda(2\cdot2+1+4)=0.$$

$$\therefore\ \lambda=-\tfrac{2}{3}.$$

Substituting this value of λ in the above equation

$$(3x+y+1)-\tfrac{2}{3}(2x-y+4)=0,$$

i.e. $\quad 3(3x+y+1)-2(2x-y+4)=0,$

which simplifies to $\quad x+y-1=0.$

It is readily seen that this line passes through $(-3, 4)$. Hence $x+y-1=0$ is the line joining the two points $(2, -1)$ and $(-3, 4)$ and meets the two given lines at their point of intersection; that is, the three lines are concurrent.

Example 4. *Find the equation to the line joining the intersection of* $2x+y-3=0$ *and* $x-3y+2=0$ *to the intersection of* $3x+2y-1=0$ *and* $x-2y+5=0$.

Any line through the intersection of the first pair of lines is $\quad (2x+y-3)+\lambda(x-3y+2)=0, \ldots\ldots\ldots(1)$

while any line through the intersection of the second pair of lines is $(3x+2y-1)+\mu(x-2y+5)=0,$ ....(2)
where λ, μ are any arbitrary constants.

If we so choose the values of λ and μ as to make the equations (1) and (2) represent the same line, then this line will be the one that passes through the intersection of the first pair of the given lines as well as the point of intersection of the second pair.

The condition that (1) and (2) may represent identical lines is that the co-efficients of like terms should be proportional, that is

$$\frac{2+\lambda}{3+\mu}=\frac{1-3\lambda}{2-2\mu}=\frac{2\lambda-3}{5\mu-1}.$$

These two equations can be solved for λ and μ, provided that no term assumes the form $\frac{0}{0}$. Solving by the usual method, we get

$$\lambda=-\tfrac{3}{5} \text{ or } \tfrac{1}{3}. \quad \mu=-1 \text{ or } 1.$$

But we note that the solution $\lambda=\frac{1}{3}$, $\mu=1$ makes the middle term of the above ratios assume the form $\frac{0}{0}$, and hence must be rejected. We are thus left with the solution $\lambda=-\frac{3}{5}$, $\mu=-1$ and these values reduce (1) and (2) to the same equation, namely. $2y=3-x.$

This is the required equation to the line joining the intersection of $2x+y-3=0$ and $x-3y+2=0$ to the intersection of $3x+2y-1=0$ and $x-2y+5=0$.

Exercise XIX

1. Find the equation to the line joining the origin to the point of intersection of the lines $2x-3y=3$, $x+3y=15$. Determine also the point of intersection of the given lines.

2. Find the equation to the line which passes through point (4, 3) and through the meet of the lines $2x+3y-7=0$, $5x-6y+8=0$.

3. Find the equation to the line through $(-1, 2)$ and concurrent with the lines $2x-3y=12$ and $7x+10y=1$.

4. Find the equation to the line having slope 5 and passing through the intersection of the lines $x+y+1=0$ and $2x-y+5=0$.

5. Find the equation to the line passing through the meet of the lines $3x-2y+3=0$, $5x+6y-2=0$ and having gradient $-1\frac{1}{2}$.

6. A st. line passes through the intersection of the lines $x-2y+4=0$ and $4x-3y+1=0$ and is inclined at 135° to the x-axis; find its equation.

7. A line concurrent with the lines $x-7y+6=0$ and $2x+3y-4=0$ makes an intercept 3 on the y-axis; find its equation.

8. A straight line passing through the intersection of the lines $2x-5y+2=0$, $3x+y-1=0$ makes an intercept -5 on the x-axis; determine the equation to the line.

9. Find the equation to the line passing through the point of intersection of the lines $3x+5y+2=0$ and $x+2y-1=0$ and making on the axes intercepts which are (i) equal in sign and magnitude (ii) equal in magnitude but opposite in sign.

10. Find the equation to the st. line passing through the intersection of the lines $9x+7y+8=0$ and $3x+4y+1=0$ and parallel to the x-axis.

11. Find the equation to the st. line passing through the meet of the lines $6x-y-2=0$ and $7x-2y+11=0$ and parallel to the y-axis.

12. Find the equation to the st. line drawn through the intersection of the lines $3x+2y=4$ and $x-y=2$ and parallel to the line $2x-y+1=0$.

13. Obtain the equation to the st. line passing through the intersection of the lines $3x+4y+6=0$, $6x+5y-9=0$ and parallel to the $x+y=7$.

14. Obtain the equation of the st. line which is $\parallel$ to the join of (1, 3), $(-2, 1)$ and passing through the meet of the lines $3x+4y=7$ and $9x-5y+8=0$.

15. Obtain the equation of the perpendicular to the line $3x+y=0$ from the point of intersection of the lines $2x+3y-7=0$ and $5x-6y+8=0$.

16. Find the equation to the st. line through the point of intersection of the lines $5x+2y-8=0$, $x-5y+2=0$ and perpendicular to $y=7x+3$.

17. Find the equation to the st. line passing through the point of intersection of the lines $3x-7y+2=0$ and $10x-10y+3=0$ and perpendicular to the latter.

18. Find the equation to the st. line passing through the point common to the lines $y-x+1=0$ and $3x+4y=12$ so as to be perpendicular to the latter.

19. Through the intersection of the lines $24y+7y=11$ and $4x+3y+1=0$ a perpendicular is drawn to the join of $(1, 2)$, $(3, -2)$, find the equation to the perpendicular.

20. Find the equation to the lines through the point of intersection of $2x+y+10=0$ and $x+y=0$ so as to be at a distance of 2 units from the origin.

21. Show that, for all values of λ. the st. line

$$(3x+y+1)+\lambda(2x-y+4)=0$$

passes through a fixed point and find the co-ordinates of the point.

22. Show that, for all values of k, the equation

$$(5x+2y-7)+k(y-3x+2)=0$$

represents a st. line passing through a fixed point equidistant from the co-ordinate axes.

23. The equations to the sides of a triangle are

$$x-y+10=0,\quad x+18=0,\quad 3x+y+34=0\,;$$

find the equations to the altitudes and the co-ordinates of the ortho-centre.

24. The sides of a triangle are the lines $x-y+1=0$, $x+2y=1$, $2y=x+1$; find the co-ordinates of the ortho-centre

25. Find the ortho-centre of the triangle formed by the lines (i) $x-y=7$, $4x-3y=29$, $y=2x-13$.
(ii) $4x+3y=5$, $5x-2y=12$, $2x-4y+3=0$.

26. Show that the following lines are concurrent:

$$x-2y-1=0,\quad x+2y-1=0,\quad x+4y-1=0.$$

27. $x+y+2=0$, $x+2y+3=0$, $3x+4y+7=0$,

28. $2x+5y=8$, $3x-4y=35$, $x-y=11$.

29. $4x+3y+2=0$, $6x+5y+6=0$ and the join of $(0, 0)$, $(2, -3)$.

30. $x-y=2$, $2x-y+1=0$, the join of $(1, -11)$, $(-5, -2)$.

31. $4x-3y+2=0$, $3x-y-6=0$, the ⊥ from $(1, 3)$ to the line $x+y+5=0$.

32. Show that, a and b being constants, the lines

$$(4x+3y-7)+a(2x+y-3)=0,$$
$$b(4x+3y-7)+(2x+y-3)=0,$$
$$24x+7x=31$$

are concurrent.

33. Find the equation to the line joining the point of intersection of the lines $3x-4y-10=0$, $2x+3y-1=0$, to the point of intersection of the lines $2x-y-4=0$, $x+y-5=0$.

34. Find the equations to the diagonals of the parallelogram formed by the lines $x+y=3$, $x+y=7$, $x-y=5$ and $x-y=1$. Also find the angle between the two diagonals.

35. Determine the condition that the three lines

$$ax+by+c=0,\quad bx+cy+a=0,\quad cx+ay+b=0$$

may be concurrent.

36. Prove that the following lines are concurrent:

$$(a+b)x+cy+d=0,$$
$$(b+c)x+ay+d=0,$$
$$(c+a)x+by+d=0.$$

37. The vertices of a triangle are (x_1, y_1), (x_2, y_2), (x_3, y_3). Show that the equations to the altitudes of the triangle are

$$x(x_2-x_3)+y(y_2-y_3)=x_1(x_2-x_3)+y_1(y_2-y_3),$$
$$x(x_3-x_1)+y(y_3-y_1)=x_2(x_3-x_1)+y_2(y_3-y_1),$$
$$x(x_1-x_2)+y(y_1-y_2)=x_3(x_1-x_2)+y_3(y_1-y_2)\,;$$

and hence prove that the altitudes of the triangle are concurrent.

38. Using the fact that the bisectors of the angles between the lines $ax+by+c=0$, $a'x+b'y+c'=0$ have equations of the form $(ax+by+c)+\lambda(a'x+b'y+c')=0$, and are equally inclined to the two given lines, deduce the equations to the bisectors.

The Circle

Though a detailed study of the circle and its properties has been reserved for Part II of the book (a forthcoming publication), we give below in a collected form a few of the important results connected with the circle and noticed in the earlier pages.

1. The equation to the circle with centre at origin and radius r is $x^2+y^2=r^2$ (Art. 23)

2. The circle with centre at (h, k) and radius r is

$$(x-h)^2+(y-k)^2=r^2. \qquad \text{(Art. 24)}$$

3. The general equation to the circle is

$$x^2+y^2+2gx+2fy+c=0.$$

In this equation,

the co-efficient of x^2 equals the co-efficent of y^2,
and the xy-term is not present.

The centre of the circle is $(-g, -f)$
and radius $\sqrt{(g^2+f^2-c)}$. (Art. 24)

4. Similarly the equation $Ax^2+Ay^2+2Gx+2Fy+C=0$ represents a circle with centre at $(-G/A, -F/A)$ and radius equal to $\sqrt{(G^2+F^2-AC)}/A$. (Art. 24)

5. The equation to the circle whose diameter is the line joining (x_1, y_1), (x_2, y_2) is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0. \qquad \text{(Art. 30, Ex. 3)}$$

6. The equation to the circum-circle of a triangle whose vertices are given is obtained as shown in Art. 24, Ex. 3.

7. The condition that the line $y=mx+c$ should be a tangent to the circle $x^2+y^2=a^2$ is that $a=c/\sqrt{(1+m^2)}$.
(Art. 69, Ex. 5)

8. In Art. 12, the method of determining the intersections of two loci is given and a number of examples are included in Ex. III. A few illustrative examples are worked below

Example 1. *Find where the line* $x+y=7$ *intersects the circle* $x^2+y^2=25$.

Solving the two as simultaneous equations, we get

$$y^2=(25-x^2)=(7-x)^2 \quad \text{or} \quad x^2-7x+12=0.$$

$$\therefore x=4 \quad \text{or} \quad x=3.$$

$$\therefore y=3 \quad \text{or} \quad y=4.$$

The solutions are (i) $x=4, y=3$ (ii) $x=3, y=4$.

Therefore the two points of intersection are **(4, 3)** and **(3, 4)**.

Corollary : The chord of the circle intercepted by the line $x+y=7$ is the distance between the two points **(4, 3)**, (3, 4) and is equal to $\sqrt{2}$.

Example 2. *Find the equation to the tangent to the circle* $x^2+y^2-6x+4y+3=0$ *at the point* (2, 1).

By substitution, we verify that (2, 1) lies on the given circle.

The centre of the circle is $(3, -2)$ so that the slope of the radius to the point (2, 1) is $\dfrac{1+2}{2-3}$ or -3.

$\therefore$ The slope of the tangent at (2, 1) is $\frac{1}{3}$.

Hence, the equation to the tangent is $\dfrac{y-1}{x-2}=\dfrac{1}{3}$,

or $3y=x+1$.

Exercise XX

1. Find the centre and radius of each of the following circles :

1. $x^2+y^2-4x-6y-3=0$.
2. $x^2+y^2-4x-2y-4=0$.

3. $x^2+y^2-2x-9y+11=0$.
4. $4x^2+4y^2+40x-12y+93=0$.
5. $x^2+y^2+8x-6y=0$,
6. $x^2+y^2-4y+1=0$.

2. Show that the st. line $3x-4y+10=0$ is a tangent to the circle $x^2+y^2=4$.

3. Show that the line $5x-12y+13=0$ touches the circle x^2+y^2+1.

4. Find the equation of the tangent to the circle

$$x^2+y^2+2x-2y-3=0 \text{ at the point } (1, 2).$$

5. Find the equation of the tangent at the origin to the circle $\quad x^2+y^2+2gx+2fy=0$.

6. Find where the following circles and st. lines intersect

1. $x^2+y^2=10$ and $x=2y$.
2. $x^2+y^2=25$ and $y=x+1$.
3. $x^2+y^2-4x-2y+1=0$ and $x+y=1$.
4. $x^2+y^2-x-7y+10=0$ and $x+3y=11$.

7. Find the length of the chord of the circle

$$x^2+y^2-6x-2y+6=0$$

intercepted by the line $y=x$.

8. Find the length of the chord of the circle

$$x^2+y^2+2x+4y-164=0$$

intercepted by the line $12y-5x+30=0$.

SUMMARY OF RESULTS

Note. For the sake of brevity, the bare results are given here omitting the conditions, explanations etc. which are expected to be supplied by the student himself.

THE POINT

1. The *x-distance* between the points (x_1, y_1), (x_2, y_2) is x_2-x_1; the *y-distance* between the same points is y_2-y_1.

2. When the origin is shifted to (h, k) the *co-ordinates* of (x, y) change to $(x-h, y-k)$.

3. *Distance* between (x_1, y_1), (x_2, y_2)

$$=\sqrt{\{(x_1-x_2)^2+(y_1-y_2)^2\}}$$
$$=\sqrt{\{(x\text{-distance})^2+(y\text{-distance})^2\}}.$$

4. *Slope* of the join of two points:

$$(y_1-y_2)/(x_1-x_2).$$

5. Co-ordinates of *middle-point* : $[\frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2)]$.

6. Co-ordinates of a *point dividing the join of two points* in the ratio $k:l$ are $\left(\dfrac{kx_2+lx_1}{k+l}, \dfrac{ky_2+ly_1}{k+l}\right)$.

7. Co-ordinates of the *centroid* ;

$$\{\tfrac{1}{3}(x_1+x_2+x_3), \tfrac{1}{3}(y_1+y_2+y_3)\}.$$

8. Area of a *triangle* with origin as a vertex :

$$\tfrac{1}{2}(x_1y_2-x_2y_1).$$

9. Area of *any* other triangle :

$$\tfrac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)].$$

THE STRAIGHT LINE

10. Equation to a st. line : *General form* : $Ax+By+C=0$.
11. *Gradient form* : $y=mx+c$.
12. *Point slope form* : $y-y_1=m(x-x_1)$.
13. *Two-point form* : $(y-y_1)/(x-x_1)=(y_1-y_2)/(x_1-x_2)$.

14. *Intercept form*: $x/a+y/b=1$.

15. *Normal form*: $x\cos\alpha+y\sin\alpha=p$.

16. *Symmetrical form*: $(x-x_1)/\cos\theta=(y-y_1)/\sin\theta=r$.

17. Condition for *parallelism*: $m=m'$, or $ab'-a'b=0$.

18. Condition for *perpendicularity*:

$$mm'=-1, \quad \text{or} \quad aa'+bb'=0.$$

19. *Angle* between two lines:

$$\tan^{-1}\frac{m-m'}{1+mm'}, \quad \text{or} \quad \tan^{-1}\frac{ab'-a'b}{aa'+bb'}.$$

20. Line through (x_1, y_1) parallel to $ax+by+c=0$:

$$a(x-x_1)+b(y-y_1)=0.$$

21. Line through (x_1, y_1) $\perp$ to $ax+by+c=0$:

$$b(x-x_1)+a(y-y_1)=0.$$

22. *Length* of perpendicular from a point to a line:

$$-(x_1\cos\alpha+y_1\sin\alpha-p) \quad \text{or} \quad -(ax_1+by_1+c_1)/\sqrt{(a^2+b^2)}.$$

23. Equation to the angular bisectors:

$$(ax+by+c)/\sqrt{(a^2+b^2)}=\pm(a'x+b'y+c')/\sqrt{(a'^2+b'^2)}.$$

24. Equation to any line through the intersection of the lines $L=0$, $L'=0$: $L+\lambda L'=0$.

25. Condition for *concurrence* of three lines:

$$\Sigma a_1(b_2c_3-b_3c_2)=0$$

THE CIRCLE

26. Circle: centre at origin, radius a: $x^2+y^2=a^2$.

27. General equation: $x^2+y^2+2gx+2fy+c=0$.

28. Centre of circle: $(-g, -f)$; radius: $\sqrt{(g^2+f^2-c)}$.

29. Condition that $y=mx+c$ may touch $x^2+y^2=a^2$;

$$c=a\sqrt{(1+m^2)} \quad \text{or} \quad c^2=a^2(1+m^2).$$

ANSWERS

EXERCISE I (Page 8)

1. On the y-axis. **2.** On the x-axis.

EXERCISE II (Page 10)

1. IV. **2.** III. **3.** II. **4.** I and IV. **5.** I and II. **6.** II and III. **7.** I. **8.** I and III. **9.** II and IV. **10.** (i) I and III. (ii) II and IV.

EXERCISE III (Page 14)

1. Line ∥ to y-axis at a distance 2 from it. **2.** x-axis. **3.** Line ∥ to the x-axis at a distance -3 from it. **4.** Line ∥ to the y-axis at a distance $-2\frac{1}{2}$ from it. **5.** The bisector of the angles YOX′, XOY′. **6, 7.** Line through the origin, passing through the first and the third quadrants and inclined to the x-axis at an angle whose tangent is 2 (in Q. 6) and $\frac{3}{4}$ (in Q. 7). **8.** Line through the origin, passing through the second and the fourth quadrants and inclined to the x-axis at an (obtuse) angle whose tangent is $-\frac{1}{5}$. **9.** Line ∥ to the y-axis at a distance a from it. **10.** Line ∥ to the x-axis at a distance $-b$ from it. **11.** Line through the origin passing through the first and the third quadrants and inclined to the x-axis at an angle whose tangent is $\frac{1}{2}$. **12.** Line through the point (1, 0) passing through the first, the fourth and the third quadrants and inclined to the x-axis at an angle of 45°. **13.** (4, 2). **14.** (1, 2). **15.** (1, 4), $(-\frac{8}{5}, -\frac{12}{5})$. **16. 1.** (3, 0), (0, 3). **2.** (0, 0). **3.** (±2, 0), (0, ±2). **4.** (±1, 0), (0, −1). **5.** (5, 0); does not cut the y-axis. **17.** $x=y$. **18.** $y=6$. **19.** $y=x\tan\theta$. **20.** $y=x\tan\theta$. **21.** (i) $x+y=s$. (ii) $2(x+y)=s$.

EXERCISE VI (Page 17)

1. (5, 2). **2.** (i) (2, −4) (ii) (5, −8) (iii) (1, −3) (iv) $(2\frac{1}{2}, 0)$ (v) (0, 0). **3.** (1, 6). **4.** (1, 1). **5.** $(-1\frac{1}{2}, -4)$. **6.** (−8, 8). **7.** (4, 5).

EXERCISE V (Page 22)

1. 5. **2.** 5. **3.** $5\sqrt{2}$. **4.** 37. **5.** 29. **6.** $\frac{1}{2}\sqrt{61}$. **7.** 2·5. **8.** 5·3. **9.** 1. **10.** 13. **11.** 4. **12.** $\sqrt{14}$. **15.** ±8. **16.** ±40. **17.** (21, 28) or (−21, −28). **18.** (i) (20, 21), (−20, −21), (21, 20), (−21, −20). (ii) (20, 21), (−20, 21), (−20, −21), (20, −21), (21, 20), (−21, 20), (−21, −20), (21, −20).

EXERCISE VI (Page 26)

1. 1. 13 2. 25 3. 17 4. $\frac{1}{2}\sqrt{113}$ 5. 2·9 6. 7 **2.** $10\sqrt{5}$
3. $20+2\sqrt{2}$ **5.** $2\sqrt{2}, \sqrt{6}, \sqrt{2}$; 90°, 60°, 30° **9.** At (2, 1) **10.** 25
11. 34 **12.** $\sqrt{34}$ each **13.** $\sqrt{10}:1$ **14** $2\sqrt{34}$, $6\sqrt{5}$ **15.** 48
16. (2, 2) **17.** (4, −5); 3 : 4 **18.** (0, 1), (−4, 9) **19.** (1, 5) **20.** 25
21. 5 **22.** (−1, 1) **23.** 1. (3, 4); 5 2. $(-\frac{1}{2}, \frac{3}{2})$; $\frac{1}{2}\sqrt{10}$ **3.** $(\frac{3}{2}, \frac{1}{2})$; $5/\sqrt{2}$
4. $(\frac{13}{14}, -\frac{17}{14})$; $\frac{5}{14}\sqrt{34}$ **24.** $(\frac{9}{2}, \frac{3}{2})$ **25.** (0, 1); $\sqrt{17}$

EXERCISE VII (Page 32)

1. $x=y$; yes **2.** 1. $x^2+y^2=4$ **2.** $x^2+y^2-8x=33$ 3. x^2+y^2+
$6x-3y-1=0$ **4.** $x^2+y^2+2x-2y-3=0$ **5.** $x^2+y^2-2x-4y-3=0$
6. $(x-3)^2+(y-2)^2=1$ or $(x-2)^2+(y-1)^2=1$ **3.** 1. (1, 2); 3
2. (−2, 3); 5 **3.** (−2, −4); 3 **4.** $(-\frac{5}{8}, 1)$; $\frac{5}{8}$
4. $x^2+y^2-4x-10y+19=0$ **6.** $x^2+y^2+2x+4y-164=0$
7. $x^2+y^2-15x+5y=0$ **8.** $x^2+y^2=9$ **9.** $x^2+y^2=16$
10. $8(x^2+y^2)+22x-52y+85=0$ **11.** $y^2=6x-9$ **12.** $x^2=15y^2$
13. $x^2+y^2=c^2-a^2$ **14.** $x^2-3y^2+6y+9=0$ **15.** $x^2+y^2=l^2/4$

EXERCISE VIII (Page 35)

1. 1 **2.** $\sqrt{3}$ 3. $-1/\sqrt{3}$ **4.** 0 **5.** ∞ **6.** 0 **7.** ∞ **8.** 0
9. ±1 **10.** 1. 0° 2. 45° 3. 135° 4. 120° 5. 26° 34′ 6. 143° 8′
7. 75° 8. 165° **11.** 1. 1 2. −2 3. 3 **4.** 0 **12.** 1. 45° 2. 0°
3. 90° **4.** 112°37′ 5. 30°.

EXERCISE IX (Page 37)

1. 1 **2.** $\frac{1}{2}$ **3.** $-\cot\frac{\theta+\phi}{2}$ **4.** $-\frac{b}{a}\cot\frac{\theta+\phi}{2}$

EXERCISE XI (Page 46)

1. 1. (3, 4) 2. (2, −1) 3. $(1, -1\frac{1}{2})$ **2.** $(4\frac{1}{2}, -1)$, $(-1, -2\frac{1}{2})$
3. $(3, 2\frac{1}{2})$, $(1\frac{1}{2}, \frac{1}{2})$ **5.** (1, −1) (−11, 7) **6.** (3, −2) **7.** $(\frac{1}{2}, 1\frac{1}{2})$
10. 1. (−3, 1) 2. (0, 0) 3. (2, 1) 4. $(-\frac{2}{3}, 1\frac{2}{3})$
11. $[\frac{1}{3}(x_1+x_2+x_3),\ \frac{1}{3}(y_1+y_2+y_3)]$ **14.** $(\frac{4}{5}, -\frac{9}{5})$, $(-\frac{11}{2}, -6)$
15. $(\frac{17}{7}, 0)$ **16.** x-axis, 1 : 2 at (5, 0); y-axis, −4 : 7 at (0, 5)
17. −3 : 1 **18.** 1 : 1, 1 : 1 **19.** 1 : 3, 1 : 1 **20.** 1 : 4, 2 : 3

EXERCISE XII (Page 55)

1. 1. 1 2. $2\frac{1}{2}$ 3. −5 4. $-2\frac{1}{2}$ 5. $5\frac{1}{2}$ 6. $-7\frac{1}{2}$ **2.** 1. 17 2. $24\frac{1}{2}$
3. 5 4. $18\frac{1}{2}$ **4.** Area = 0; the points are collinear **6.** 10
7. 1. $y=2x+3$ 2. $y=2x-1$ **8.** $x+y=9$ **9.** $y=x+3$ or $y=x-5$
10. Area $=\frac{1}{2}[x(y_1-y_2)-y(x_1-x_2)+(x_1y_2-x_2y_1)]$; this expression equated to 0 is the equation to the st. line. **11.** $x/a+y/b=2$[illegible]
13. 3 : 2 **14.** 1 : 3 **15.** 2 : 3, 1 : 4 **16.** 2 : 3, 2 : 1; (2, −1) **17.** 3 : 1, 2 : 4

EXERCISE XIII (Page 67)

1. 1. $y=2x+3$ 2. $4y=3x-8$ 3. $y=\sqrt{3}x-4$ 4. $x+y=0$ 5. $3x+y=0$
6. $12y=5x+60$ **2,** 1. 4, 7 2. 3, −9 3. −5, 11 4. $-3\frac{1}{2}$, −5
5. $-a/b$, c/b **6.** $1/l$, $-m/l$ **3.** 1. $c=5$, 2. $a/l=b/m=c/n$, 3. $am+b=0$,
$b=c$ 4. $p+mq=0$, $r+cq=0$ 5. $p\sqrt{(a^2+b^2)}=-c$ 6. $l/3=m/4=n/6$
4. 1. $y=3x$ 2. $4x+3y=11$ 3. $y+5=0$ 4. $y=x+4$
5. $y+\sqrt{3}x=7-4\sqrt{3}$ 6. $x+\sqrt{3}y=\sqrt{3}$ 7. $7x+y+3=0$ 8. $y=7x-15$
5. 1. $4x=3y$ 2. $2x+y=0$ 3. $y=x+1$ 4. $x=y+2$ 5. $3x+y=4$
6. $12x+4y+7=0$ 7. $x+y=a+b$ 8. $y(t_1+t_2)=2x+2at_1t_2$
9. $x\cos\frac{\theta+\phi}{2}+y\sin\frac{\theta+\phi}{2}=a\cos\frac{\theta-\phi}{2}$

10. $\frac{x}{a}\cos\frac{\theta_1+\theta_2}{2}+\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}=\cos\frac{\theta_1-\theta_2}{2}$

6. 1. $y=4x-13$, $3x+2y=7$, $3y=x+5$ 2. $4y=x-15$, $y=3x-1$,
$8x+y=21$ **7.** $3x+10y-2=0$; x-axis at $(\frac{2}{3}, 0)$; y-axis at $(0, \frac{1}{5})$
8. 1. $y=1$, $9x+y=7$, $9x+8y=14$ 2. $5y=x+11$, $4x+13y=33$,
$5x+8y=22$ **9.** Sides: $x+y=9$, $5y=8x-33$, $y=12x-17$
Medians: $3y=10x-25$, $16x+3y=53$, $9y=4x+3$

EXERCISE XIV (Page 77)

1. 1. ‖ 2. ⊥ 3. ⊥ 4. ‖ 5. $\tan^{-1}\frac{33}{56}$ 6. ⊥ 7. ‖ 8. $\tan^{-1}\frac{1}{8}$
9. $\tan^{-1}\frac{2}{25}$ **6.** $\tan^{-1}\frac{1}{3}$ **7.** $y=2$ **8.** $2y=x+2$ **9.** $5x+3y+11=0$
10. $x+y=8$ **11.** $4x-7y+43=0$ **12.** $7x+3y=30$ **13.** $4y=x+18$
14. $2x+5y+11=0$ **15.** $2x-12y-9=0$; $(4\frac{1}{2}; 0)$, $(0, -\frac{3}{4})$
16. $9x+2y-8=0$ **17.** $y=(2+\sqrt{3})x$, $y=(2-\sqrt{3})x$ **19.** $3x+y=12$,
$3y=x+16$ **20.** 1·5, 5, 1; 56° 19′, 78° 41′, 45° **21.** $9x+4y=29$
22. $2x+y=6$; (3, 0), (0, 6) **23.** $9x+4y+28=0$ **24.** $5y=6x-16$
25. $3x+y=0$, $7y=2x+31$, $x+8y=31$ **26.** $x+3y=11$, $5y=x+15$;
$(1\frac{1}{4}, 3\frac{1}{4})$ **27.** $(\frac{1}{3}, \frac{11}{3})$ **29.** (−4, 1) **30.** $(\frac{29}{11}, \frac{35}{11})$

EXERCISE XV (Page 83)

1. $2x+5y=10$ **2.** $3y=5x+10$ **3.** $4x+6y+3=0$ **4.** $\sqrt{3}y=x+6$
5. $x\cos\alpha+y\sin\alpha=p$ **6.** $y=3x-6$ **7.** $5x+12y+30=0$
8. (i) $x+y=5$ (ii) $y=x+1$ **9.** $x+3y+16=0$ **10.** $7x-4y+37=0$
11. $5x+3y=30$ **12.** $12x+9y+96=0$ **13.** $3x-2y+4=0$
14. $(x_1y_2-x_2y_1)/(y_2-y_1)$, $(x_1y_2-x_2y_1)/(x_1-x_2)$.

EXERCISE XVI (Page 92)

1. $x+y=4\sqrt{2}$ **2.** $x\sqrt{3}+y+6=0$ **3.** $\sqrt{3}x-3y-6=0$ **4.** $4x+3y=11$
5. $\frac{3}{5}x+\frac{4}{5}y=2$; $+2$ **6.** $-\frac{12}{13}x+\frac{5}{13}y=3$; $+3$ **7.** $\frac{15}{17}x+\frac{8}{17}y=3$; $+3$
8. $-\frac{9}{41}x+\frac{40}{41}y=-\frac{2}{5}$; $-\frac{2}{5}$ **9.** 1. $(x-4)\cos\alpha+(y-7)\sin\alpha=0$
2. $(x+2)\cos\alpha+(y+3)\sin\alpha=0$ 3. $x\cos\alpha+y\sin\alpha+p=0$
10. 1. 4 2. 2 3. 4 4. 1 5. $\sqrt{13}/2$ **13.** $x^2+y^2=9$

EXERCISE XVII (Page 100)

1. -4 **2.** $+3$ **3.** $+1$ **4.** $-\frac{1}{2}$ **5.** $+\sqrt{10}$ **6.** $2\sqrt{2}$ **7.** $2\frac{1}{5}$; origin side **8.** 2; non-origin side **9.** $-2/\sqrt{5}$; origin side **10.** -2; origin side **11.** $-1/\sqrt{5}$, $-1/\sqrt{5}$; on the same side; the join of the points is ∥ to the given line **12.** $1/\sqrt{41}$, $3/\sqrt{41}$; on the same side **15.** $5x^2+10xy+5y^2+2x-2y-13=0$ **16.** $x^2+2xy+y^2-22x-6y+25=0$

EXERCISE XVIII (Page 105)

1. $y+1=0$, $3x+5=0$ **2.** $x+y=3$, $x-y+1=0$ **3.** $x-2y+1=0$, $2x+y=3$ **4.** $x+3y-11=0$, $3x-y+7=0$ **5.** $3x+11y-4=0$, $99x-27+128=0$ **6.** $x+y=2$, $3y=3x+5$ **8.** $(\frac{5}{8}, -\frac{5}{4})$ **9.** $(-\frac{28}{9}, -\frac{16}{9})$
11. $-4\frac{1}{3}$

EXERCISE XIX (Page 112)

1. $x=2y$, (6, 3) **2.** $3y=x+5$ **3.** $x+y=1$ **4.** $y=5x+11$ **5.** $3x+2y=0$
6. $x+y=5$ **7.** $7x+2y=6$ **8.** $11y=x+5$ **9.** $x+y+4=0$, $x-y+14=0$
10. $y-1=0$ **11.** $x=3$ **12.** $10x=5y+18$ **13.** $3x+3y+5=0$
14. $3y=2x+5$ **15.** $x-3y+5=0$ **16.** $x+7y=6$ **17.** $4x+4y=1$
18. $28x-29y=37$ **19.** $x-2y=4$ **20.** $3x+4y=10$, $4x+3y+10=0$
21. $(-1, 2)$ **23.** $x+y=2$, $y+1=0$, $3y=x-6$; $(3, -1)$ **24.** $(-\frac{1}{2}, 1)$
25. (i) $(8, -6)$ (ii) $(1\frac{1}{8}, \frac{3}{4})$ **33.** $y=3x-7$ **34.** $x=4$, $y=1$, angle$=90°$

EXERCISE XX (on the Circle) (Page 117)

1. 1. (2, 3), 4 2. (2, 1), 3 3. $(1, 4\frac{1}{2})$, $\sqrt{10\frac{1}{4}}$ 4. $(-5, 1\frac{1}{2})$ 2
5. $(-4, 3)$, 5 6. (0, 2), $\sqrt{3}$ **4.** $2x+y-4=0$ **5.** $gx+fy=0$
6. 1. $(2\sqrt{2}, \sqrt{2})$, $(-2\sqrt{2}, -\sqrt{2})$ 2. $(-4, -3)$, (3, 4) 3. (2, 3), (4, 1) 4. (2, 3) $(4, -1)$ **7.** $2\sqrt{2}$ **8.** 26

MYSORE UNIVERSITY

INTERMEDIATE EXAMINATION PAPERS

1942

1. a) Prove that the points (12, 0), (0, 9), $(-6, 0)$ and $(0, -8)$ are concyclic and find the radius of the circle through them.

   b) Obtain the co-ordinates of the reflection of the point (2, 3) in the line $6x+4y=11$.

2. Find the equation of a st. line passing through the point (x_1, y_1) and having the slope m.

   Obtain the equation of the st. line passing through the point of intersection of the lines $2x-3y=1$ and $x+y=3$ and inclined at 45° to the $+ve$ direction of the x-axis. Show also that this st. line passes through the centre of the circle $x^2+y^2-4x-2y-20=0$.

1943

1. Find the length of the perpendicular from the point (x_1, y_1), on the st. line $ax+by+c=0$. Find also the equation to the perpendicular. Find the co-ordinates of the ortho-centre of the triangle formed by the origin and the points A(3, 2) and B(2, 3). Show also that the ortho-centre divides the perpendicular from O on AB in the ratio 24 : 1.

2. a) Find the equation of the tangent at the origin to the circle $x^2+y^2+2gx+2fy=0$.

   b) Find the length of the chord of the circle $x^2+y^2-6x-2y+6=0$ intercepted by the line $y=x$.

1944

1. Find the angle between the st. lines $x-2y+3=0$; $2x+3y-8=0$. Also find (i) the area of the triangle

included between these lines and the x-axis, (ii) the co-ordinates of the centroid of this triangle.

2. Find the length of the perpendicular from the point (x_1, y_1) on the st line $ax+by+c=0$. Show that the st. lines $3x-4y+10=0$ and $5x-12y+13=0$ are respectively tangents to the circles $x^2+y^2=4$ and $x^2+y^2=1$ and that $14x+8y+65=0$ is the bisector of one of the angles between the st. lines.

1945

1. a) Find the locus of a point which moves such that the area of the triangle formed by it with the two points (2, 5) and (–4, 3) is 14 square units.

b) Find the equation of the st. line passing through the intersection of $x-y+1=0$ and $4x+3y=12$ so as to be perpendicular to the latter.

2. a) Prove that the four points (0, –2), (5, 1), (2, 6) and (–3, 3) form the vertices of a square; write down the equations to the sides and the diagonals.

b) Find the equation to the circle with centre O and touching the line $8x+15y=34$.

1946

1. a) Obtain the co-ordinates of the point dividing the join of (x_1, y_1) and (x_2, y_2) in the ratio $p:q$.

b) The vertices of a quadrilateral ABCD taken in order are (–1, –2), (2, 1), (3, 4) and (1, 2). Find the ratio in which AC divides BD; find also the equations of AC and BD.

2. a) Obtain the equation of a st. line in the form

$$x \cos \alpha + y \sin \alpha = p.$$

b) Find the distance between $3x-4y+5=0$ and $3x-4y-8=0$. Show that the latter line touches the circle $x^2+y^2+6x-4y-12=0$; obtain the co-ordinates of the point of contact.

1947

1. Obtain an expression for the perpendicular distance from the point (x_1, y_1) to the line $ax+by+c=0$.

A square has two opposite vertices at the points (–5, 8) and (7, 13). Find the co-ordinates of the other vertices and the equations of the sides and diagonals.

2. a) Show that $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$ represents a circle of which (x_1, y_1), (x_2, y_2) are a pair of diametrically opposite points.

b) Find the equations of two equal circles which touch each other and have their centres at (4, 5) and (7,6)

1948

1. a) Obtain the equation to the locus of a point whose distance from the point (2, –3) is equal to its distance from the line $x+4=0$.

b) Show that C(4, –3) is a point on the line joining A(1, 3) and B(5, –5) and find the co-ordinates of the point D which divides AB externally in the ratio AC : CB

2. a) Find the angle between the st. lines $ax+by+c=0$ and $a'x+b'y+c'=0$ and deduce the conditions that the lines may be (i) parallel
(ii) perpendicular.

b) Show that the st. line $12y=5x-26$ touches the circle $x^2+y^2+14x-18y-39=0$ and find the co-ordinates of the point of contact.

MYSORE UNIVERSITY

INTERMEDIATE EXAMINATION PAPERS

1942

1. a) $\frac{5\sqrt{3}}{2}$ b) $(-1, 1)$ 2. $y-y_1=m(x-x_1)$; $x-y-1=0$

1943

1. $-(ax_1+by_1+c)/\sqrt{a^2+b^2}$; $bx-ay=bx_1-ay_1$; $(\frac{12}{5}, \frac{12}{5})$
2. (a) $gx+fy=0$ (b) $2\sqrt{2}$.

1944

1. $\tan^{-1}(\frac{7}{4})$; 7; $(\frac{2}{3}, \frac{2}{3})$ 2. $-(ax_1+by_1+c)/\sqrt{a^2+b^2}$.

1945

1. a) $x-3y=1$. b) $21x-28y=37$. 2. a) $3x-5y=10$, $5x+3y=28$, $3x-5y+24=0$, $5x+3y+6=0$, $4x-y=2$, $x+4y=9$.
(b) $x^2+y^9=4$.

1946

1. a) $\left[\frac{px_2+qx_1}{p+q}, \frac{py_2+qy_1}{p+q}\right]$ b) 2 : 3; $3x-2y=1$, $x+y=3$.
2. b) $\frac{13}{5}$; $(0, -2)$

1947

1. $-(ax_1+by_1+c)/\sqrt{a^2+b^2}$; $(\frac{7}{2}, \frac{9}{2})$ and $(-\frac{3}{2}, \frac{33}{2})$, $7x+17y=101$, $17x-7y-28=0$, $7x+17y-270=0$, $17x-7y+141=0$;
$5x-12y+121=0$, $24x+10y=129$
2. b) $(x-4)^2+(y-5)^2=\frac{5}{2}$, $(x-7)^2+(y-6)^2=\frac{5}{2}$.

1948

1. a) The locus is the curve given by $y^2+6y-12x-3=0$
b) $(7, -9)$ 2. a) $\tan^{-1}\left(\frac{ab^1+a^1b}{aa^1+bb^1}\right)$, $ab^1-a^1b=0$, $aa^1+bb^1=0$
b) $(-2, -3)$.

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